Electronics and Signal Processing

Lecture Notes

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November 3, 2026

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Sources and Attribution

The end-of-chapter problems in these notes are taken from the following openly licensed textbooks. Problem numbers in the text (for example 5.53) are the numbers of the end-of-chapter problems in the book named in the problems section.

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- OpenStax, *University Physics Volume 1*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-1
- OpenStax, *Calculus Volume 3*, by Gilbert Strang, Edwin "Jed" Herman et al. Licensed under CC BY 4.0. https://openstax.org/details/books/calculus-volume-3

The signal-processing chapter also draws on the following book, which is not openly licensed and is cited rather than reproduced: Marina Bosi and Richard E. Goldberg, *Introduction to Digital Audio Coding and Standards*, Kluwer Academic Publishers, 2003. The approximation of the threshold in quiet used there is taken from that book; the plot of it is our own.

The Fourier problems at the end of the Fourier chapter were written for this course by the author.

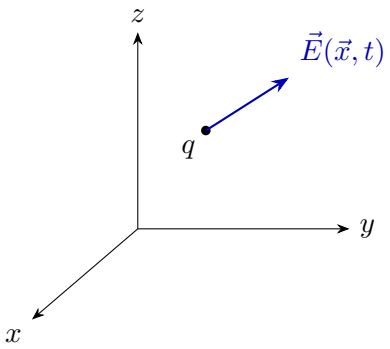
The licence is Creative Commons Attribution 4.0 International (https://creativecommons.org/licenses/by/4.0/). Changes: figures have been redrawn, and the wording of some problems has been lightly edited (for example "as shown below" → "as shown"), and the brief answers are taken from the answer keys provided by OpenStax. The original problems, figures and answers are available at the sources above. These notes are not endorsed by OpenStax.

Chapter 1

Electric Fields & Capacitance

1.1 Electric field

The **electric field** $\vec{E}(\vec{x}, t)$ maps a point $\vec{x}$ and a time t to a vector $\vec{E}$.



Physically: $\vec{E}$ tells us how a test charge will move when placed at $\vec{x}$. The force on a test particle of charge q is

$$\boxed{\vec{F} = q\,\vec{E}} \quad \Longleftrightarrow \quad \vec{E} = \frac{\vec{F}}{q}. \tag{1.1}$$

Electric charge

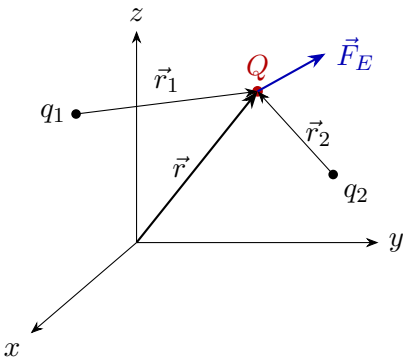
Electric charge plays two roles:

1. it is the propensity for *producing* an electric field;
2. it determines the force, $\vec{F}$ in *response* to an applied electric field. From (1.1), $q = |\vec{F}|/|\vec{E}|$.

Charges, at rest or in motion, are the sources of *all* electromagnetic fields.

1.2 Coulomb's law

Coulomb's law gives the electric force $\vec{F}_E$ that *static* point charges (i.e. in vacuum, at rest) exert on other charges.



$$\boxed{\vec{F}_E(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\,\hat{r}_i} \tag{1.2}$$

where

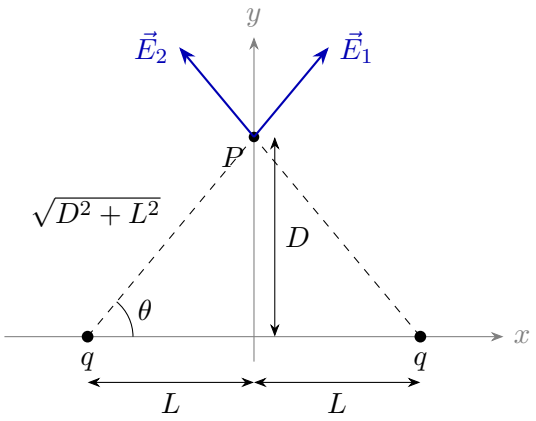
- Q is the charge of the particle experiencing $\vec{F}_E$;
- $\vec{r}_i$ is the separation vector from charge q_i to Q, with length $r_i = |\vec{r}_i|$ and unit vector $\hat{r}_i = \dfrac{\vec{r}_i}{|\vec{r}_i|}$;
- ε_0 is the permittivity of free space, $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$.

Combining (1.1) and (1.2) gives the electric field produced by N charged particles:

$$\boxed{\vec{E}(\vec{r}) = \frac{\vec{F}_E}{Q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\,\hat{r}_i}$$

Example: two equal charges

Two charges q sit on the x-axis at $x = \pm L$. Find the field at point P, a height D above their midpoint.



By superposition, $\vec{E}(P) = \vec{E}_1 + \vec{E}_2 = (E_x^1 + E_x^2)\,\hat{\imath} + (E_y^1 + E_y^2)\,\hat{\jmath}$. By symmetry $E_x^2 = -E_x^1$ and $E_y^2 = E_y^1$, so

$$\vec{E}(P) = 2E_y^1\,\hat{\jmath}.$$

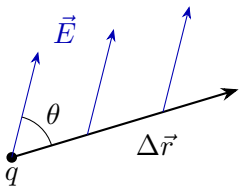
With $\sin\theta = D/\sqrt{D^2+L^2}$ and $r_1^2 = D^2 + L^2$,

$$E_y^1 = \frac{q}{4\pi\varepsilon_0}\frac{\sin\theta}{r_1^2} = \frac{q}{4\pi\varepsilon_0}\frac{D}{(D^2+L^2)^{3/2}},$$

and therefore

$$\boxed{\vec{E}(P) = \frac{q}{2\pi\varepsilon_0}\frac{D}{(D^2+L^2)^{3/2}}\,\hat{\jmath}}$$

1.3 Voltage



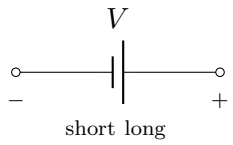
$|\Delta\vec{r}| = d$

Work done by the electric force on q as it moves through $\Delta\vec{r}$ in a uniform field:

$$W_E = \vec{F}_E \cdot \Delta\vec{r} = q\,E\,d\cos\theta.$$

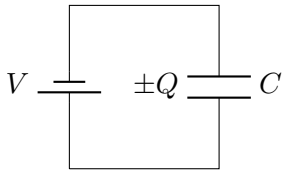
The **electric potential** difference, a.k.a. "voltage", is the (negative of the) work done per unit charge:

$$\boxed{\Delta V = -\frac{W_E}{q} = -E\,d\cos\theta} \tag{1.3}$$



A **battery** is a device that maintains a potential difference V between two conducting terminals. Its circuit symbol has two parallel lines of unequal length: the *longer* line is the positive terminal (higher potential) and the *shorter* line is the negative terminal (lower potential), so $V = V_+ - V_-$. We'll use this symbol for any constant voltage source.

1.4 Capacitors



A capacitor is made of two conducting plates across which a voltage V can be applied. One plate acquires charge $+Q$, the other $-Q$.

Capacitance: $\boxed{C = \dfrac{Q}{V} = \dfrac{\text{charge on plate}}{\text{voltage}}}$

Units: farad, F = C/V.

Parallel-plate capacitor

For a parallel-plate capacitor with plate area A and separation d, the field between the plates is uniform:

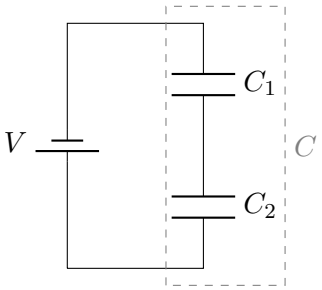
$$E = \frac{Q}{\varepsilon_0 A}.$$

Using (1.3) (magnitude, with $\theta = 0$), $V = Ed$, so

$$C_\parallel = \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q}{\frac{Q}{\varepsilon_0 A}\,d} \quad\Longrightarrow\quad \boxed{C_\parallel = \frac{\varepsilon_0 A}{d}}$$

Note that Q cancels: capacitance depends only on geometry (a general result).

Capacitors in series



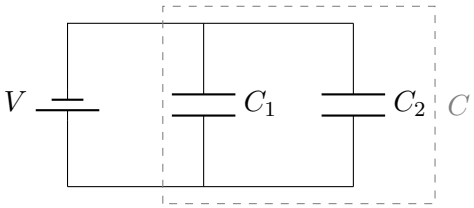
Let C be the effective capacitance of the combination.

- Conservation of energy (Kirchhoff's voltage law): $V_1 + V_2 = V$.
- Conservation of charge: the inner plates carry equal and opposite charge, $Q_1^+ = -Q_2^-$, so both capacitors hold the same Q.

$$C_1 = \frac{Q}{V_1}, \qquad C_2 = \frac{Q}{V_2} \qquad\Longrightarrow\qquad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{C_1} + \frac{Q}{C_2}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

Capacitors in parallel



Here the charges add and the voltages are equal:

$$Q = Q_1 + Q_2, \qquad V_1 = V_2 = V.$$

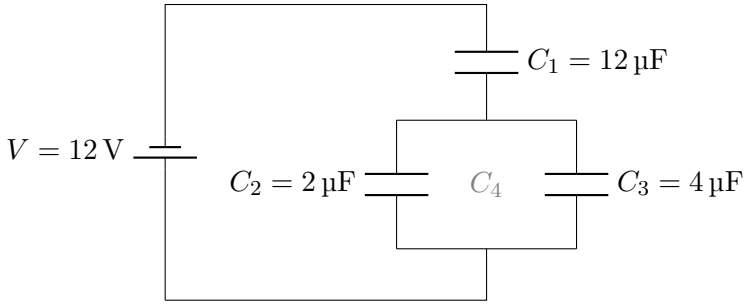
So

$$C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = C_1 + C_2,$$

$$\boxed{C = C_1 + C_2}$$

Example

Find the effective capacitance of the circuit below.



C_2 and C_3 are in parallel, so they combine to $C_4 = C_2 + C_3 = 6\,\mu\text{F}$. C_4 is then in series with C_1:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_4} \implies C = \frac{C_1 C_4}{C_1 + C_4} = \frac{12 \cdot 6}{12 + 6}\,\mu\text{F} \qquad \implies \qquad \boxed{C = 4\,\mu\text{F}}$$

Problems

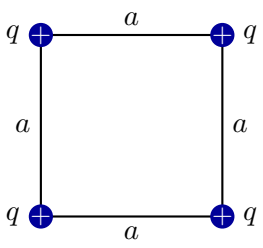
Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax University Physics Vol. 2, Chapter 5: Electric Charges and Fields

5.53 Point charges $q_1 = 50\,\mu\text{C}$ and $q_2 = -25\,\mu\text{C}$ are placed 1.0 m apart. What is the force on a third charge $q_3 = 20\,\mu\text{C}$ placed midway between q_1 and q_2?

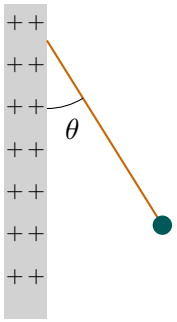
[Answer →]

5.63 What is the force on the charge q at the lower-right-hand corner of the square shown here?



[Answer →]

5.95 Shown is a small sphere of mass 0.25 g that carries a charge of 9.0×10^{-10} C. The sphere is attached to one end of a very thin silk string 5.0 cm long. The other end of the string is attached to a large vertical conducting plate that has a charge density of 30×10^{-6} C/m^2. What is the angle that the string makes with the vertical?



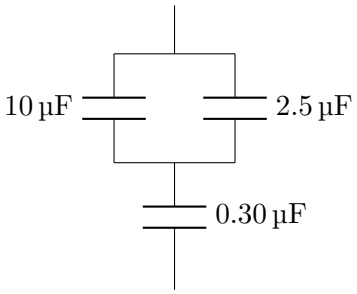
[Answer →]

OpenStax University Physics Vol. 2, Chapter 8: Capacitance

8.27 A set of parallel plates has a capacitance of 5.0 µF. How much charge must be added to the plates to increase the potential difference between them by 100 V?

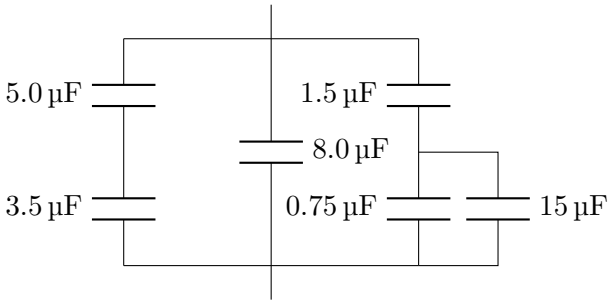
[Answer →]

8.33 Find the total capacitance of this combination of series and parallel capacitors.



[Answer →]

8.37 Find the net capacitance of the combination of series and parallel capacitors shown.

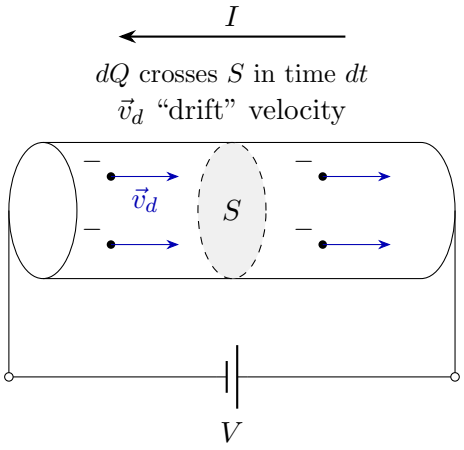


[Answer →]

Chapter 2

Current, Resistance & DC Circuits

2.1 Current and Ohm's law



Current: let dQ be the charge that passes through the surface S (the wire's cross-section) in a time dt. Then

$$\boxed{I := \frac{dQ}{dt}}$$

measured in amperes, A = C/s (coulombs per second). A voltage source attached across the wire drives the current.

Ohm's law

Inside a conductor, the electric field drives a current. The local **current density** $\vec{J}$ (current per unit area, in A/m^2) is proportional to the field, and the factor of proportionality is the **resistivity** ρ, a property of the material:

$$\vec{E} = \rho\, \vec{J}.$$

Now consider a uniform wire of length L and cross-sectional area A_S (the area of S), with a voltage V across its ends. The field is uniform, so $E = V/L$, and the current density is uniform, so $I = J\, A_S$. Then

$$V = E\, L = \rho\, J\, L = \rho\, \frac{I}{A_S}\, L,$$

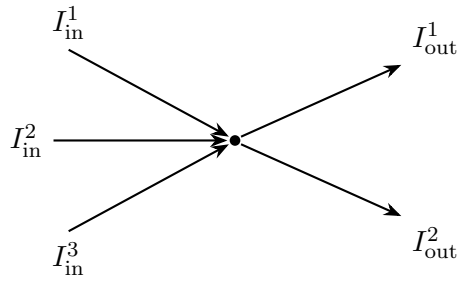
that is,

$$\boxed{V = R\, I}, \qquad \boxed{R = \frac{\rho L}{A_S}}. \tag{2.1}$$

This is **Ohm's law**: the voltage across the wire is proportional to the current through it, where the **resistance** R is a constant that depends on the electrical (through ρ) and geometric (L, A_S) properties of the wire. Resistance is measured in ohms, Ω = V/A.

2.2 Kirchhoff's rules

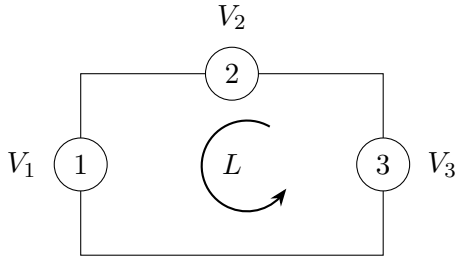
1. Junction rule



$$\sum_{i=1}^{N} I^i_{\text{in}} = \sum_{i=1}^{M} I^i_{\text{out}}$$

The total current into a junction equals the total current out. This is *conservation of charge*.

2. Loop rule

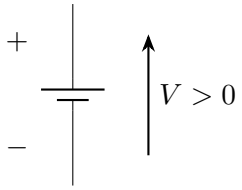


$$\sum_{\text{closed loop}} V_i = 0$$

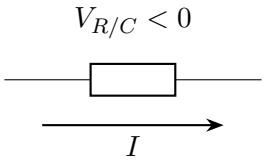
The voltages around any closed loop sum to zero. This is *conservation of energy*.

To apply the loop rule:

- Choose a loop, with a direction.
- Use sign conventions, e.g.:



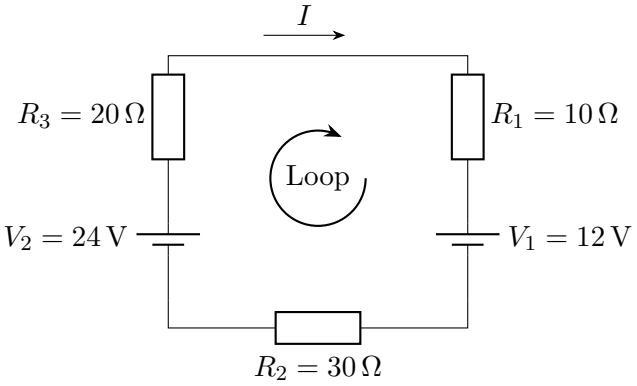
A battery contributes the voltage it induces, $V > 0$ (traversed from its $-$ to its $+$ terminal).



A resistor or capacitor (traversed along the current I) gives a voltage drop over the component, $V_{R/C} < 0$.

Example (OpenStax Example 10.7)

Find I.

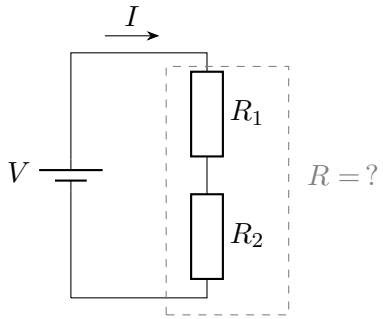


The loop rule (watch the signs!) gives

$$V_2 - IR_3 - IR_1 - V_1 - IR_2 = 0 \implies \boxed{I = \frac{V_2 - V_1}{R_3 + R_2 + R_1} = \frac{12\,\text{V}}{60\,\Omega} = 0.2\,\text{A}}$$

2.3 Resistors in series and parallel

Series

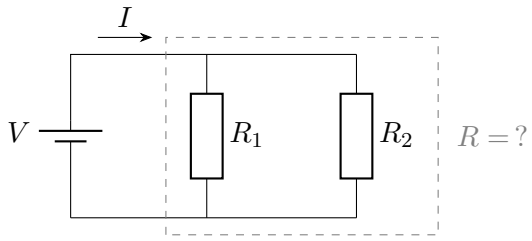


Let R be the effective resistance of the combination. By the junction rule, the same current flows through both resistors, $I_1 = I_2 = I$. By the loop rule, $V = V_1 + V_2$, so by (2.1)

$$R = \frac{V}{I} = \frac{R_1 I + R_2 I}{I}$$

$$\boxed{R = R_1 + R_2}$$

Parallel



Here the junction rule gives $I = I_1 + I_2$, and the loop rule gives equal voltages across the two resistors:

$$I = I_1 + I_2, \qquad V = V_1 = V_2.$$

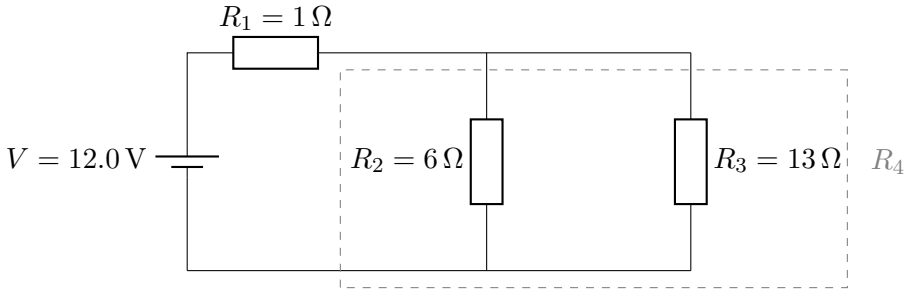
By Ohm's law,

$$\frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}}$$

Example (OpenStax Example 10.4)

Find the equivalent resistance R of the circuit below.



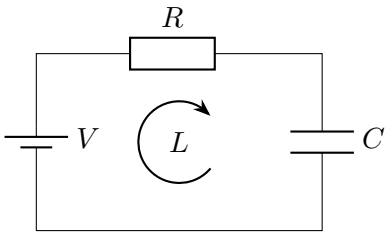
First find R_4, the combination of R_2 and R_3 in parallel:

$$\frac{1}{R_4} = \frac{1}{R_2} + \frac{1}{R_3} \implies R_4 = \frac{R_2 R_3}{R_2 + R_3} = \frac{78}{19}\,\Omega.$$

Then R_4 is in series with R_1:

$$\boxed{R = R_1 + R_4 = \left(1 + \frac{78}{19}\right)\Omega = 5.10\,\Omega}$$

2.4 RC circuits



Loop rule around the loop L:

$$V - V_R - V_C = 0,$$

with $V_R = IR$ (Ohm's law) and $V_C = Q/C$ (capacitance). Hence $V - IR - Q/C = 0$, and with $I = dQ/dt$,

$$\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)} \tag{2.2}$$

This is a differential equation for $Q(t)$.

Charging a capacitor

Take a constant applied voltage and no initial charge:

$$V(t) = V, \qquad Q(0) = 0.$$

Then (2.2) reads $RC\,\dfrac{dQ}{dt} = CV - Q$, i.e.

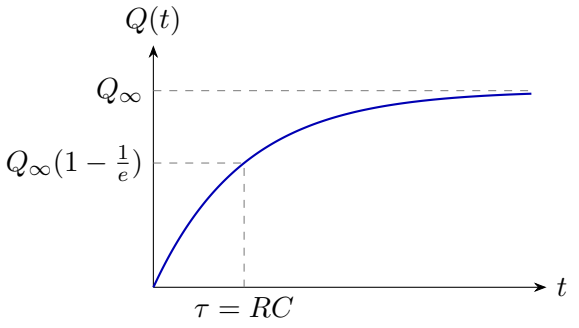
$$\frac{dQ}{CV - Q} = \frac{dt}{RC}.$$

Integrating, with the substitution $u = CV - Q$, $du = -dQ$:

$$\int_0^{Q(t)} \frac{dQ}{CV - Q} = \frac{t}{RC} \quad \Longrightarrow \quad -\int_{CV}^{CV-Q(t)} \frac{du}{u} = \ln\frac{CV}{CV - Q(t)} = \frac{t}{RC}.$$

Therefore $e^{t/RC} = \dfrac{CV}{CV - Q(t)}$, i.e.

$$\boxed{Q(t) = CV\left(1 - e^{-t/RC}\right)}$$



Call

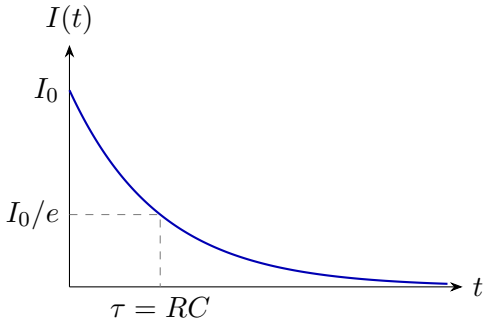
$$Q_\infty := CV, \qquad \tau := RC,$$

the final charge and the charging/decay time.

The current is

$$\boxed{I(t) = \frac{dQ}{dt} = \frac{CV}{RC}\,e^{-t/RC} = \frac{V}{R}\,e^{-t/RC}}$$

with $I_0 := \dfrac{V}{R}$.



Discharging a capacitor

Now remove the source and start from a charged capacitor:

$$V(t) = 0, \qquad Q(0) = CV_0.$$

Then (2.2) becomes

$$\frac{dQ}{dt} = -\frac{1}{RC}\,Q \quad \Longrightarrow \quad -\frac{dQ}{Q} = \frac{dt}{RC}. \tag{2.3}$$

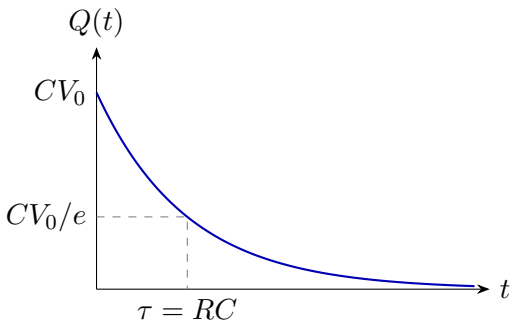
Integrating,

$$-\int_{CV_0}^{Q(t)} \frac{dQ}{Q} = \frac{t}{RC} \quad \Longrightarrow \quad \ln\frac{Q(t)}{CV_0} = -\frac{t}{RC},$$

so

$$\boxed{Q(t) = CV_0\, e^{-t/RC}}$$

$$\boxed{I(t) = \frac{dQ}{dt} = -\frac{V_0}{R}\, e^{-t/RC}}$$



Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax University Physics Vol. 2, Chapter 9: Current and Resistance

9.27 The charge on a capacitor in a circuit is modeled as $Q(t) = Q_{\max}\cos(\omega t + \phi)$. What is the current through the circuit as a function of time?

[Answer →]

9.37 What is the resistance of a 20.0 m-long piece of 12-gauge copper wire having a 2.053 mm diameter?

[Answer →]

9.51 A resistor rated at 250 kΩ is connected across two D cell batteries (each 1.50 V) in series, with a total voltage of 3.00 V. The manufacturer advertises that their resistors are within 5% of the rated value. What are the possible minimum current and maximum current through the resistor?

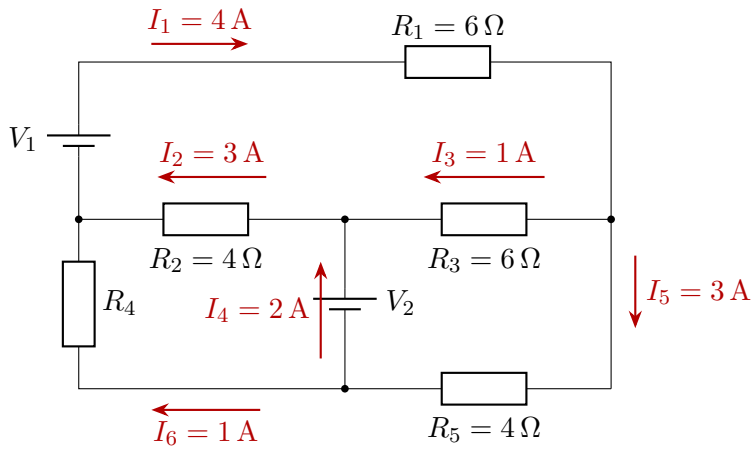
[Answer →]

OpenStax University Physics Vol. 2, Chapter 10: Direct-Current Circuits

10.31 Referring to the example combining series and parallel circuits in this lecture (Example 10.4 above), calculate I_3 in the following two different ways: (a) from the known values of I and I_2; (b) using Ohm's law for R_3. In both parts, explicitly show how you follow the steps in the problem-solving strategy for series and parallel resistors.

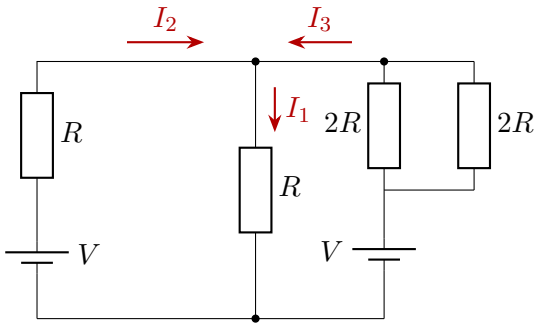
[Answer →]

10.39 Consider the circuit shown. Find V_1, V_2, and R_4.



[Answer →]

10.43 Consider the circuit shown. Write equations for the three currents in terms of R and V.



[Answer →]

10.53 A $500\,\Omega$ resistor, an uncharged $1.50\,\mu\text{F}$ capacitor, and a $6.16\,\text{V}$ emf are connected in series. (a) What is the initial current? (b) What is the RC time constant? (c) What is the current after one time constant? (d) What is the voltage on the capacitor after one time constant?

[Answer →]

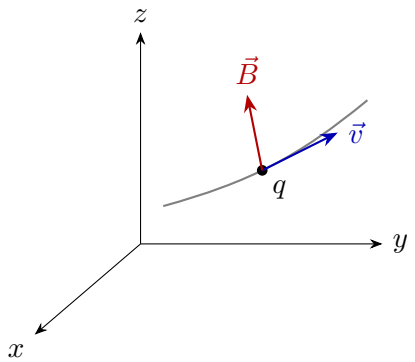
10.57 If you wish to take a picture of a bullet traveling at $500\,\text{m/s}$, then a very brief flash of light produced by an RC discharge through a flash tube can limit blurring. Assuming $1.00\,\text{mm}$ of motion during one RC constant is acceptable, and given that the flash is driven by a $600\,\mu\text{F}$ capacitor, what is the resistance in the flash tube?

[Answer →]

Chapter 3

Magnetic Fields and Inductors

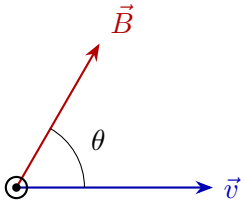
3.1 Magnetic force



Magnetic force arises when charges move. The force on a charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\boxed{\vec{F}_m = q\,\vec{v} \times \vec{B}} \tag{3.1}$$

This is the **Lorentz force**.



$\vec{F}_m$ (out of page)

Its magnitude is

$$\boxed{F_m = q\,v\,B\sin\theta}$$

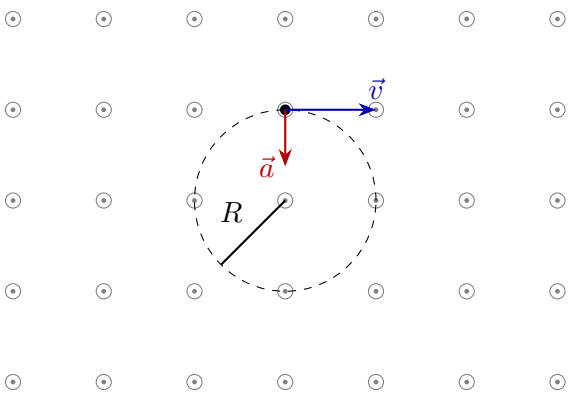
where θ is the angle between $\vec{v}$ and $\vec{B}$. The direction is given by the right-hand rule (RHR).

Note: $\vec{F}_m \perp \vec{v}$, so the magnetic force is a *deflecting* force.

Special case: circular motion

If $\vec{v} \perp \vec{B}$ then $\theta = \pi/2$ and

$$\boxed{F_m = q\,v\,B}$$

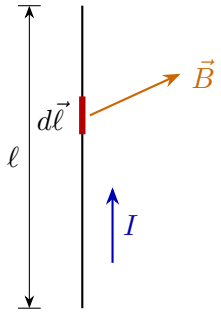


$\odot$: $\vec{B}$ out of page

Newton's second law, $\vec{F}_m = m\vec{a}$, with circular motion, $a = v^2/R$, gives $F_m = mv^2/R$. Hence

$$\frac{mv^2}{R} = q\,v\,B \quad \Longrightarrow \quad \boxed{R = \frac{m\,v}{q\,B}}$$

Force on a current-carrying wire



The force on a wire in a field $\vec{B}$ follows from summing the Lorentz force over the moving charges. For a small element $d\vec{\ell}$ of the wire, where $\vec{\ell}$ is a vector pointing along the wire,

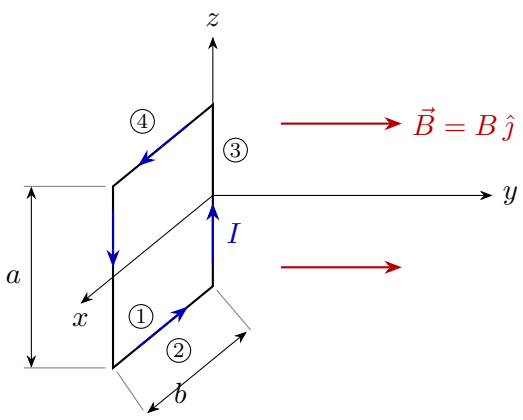
$$d\vec{F}_m = dQ\,\frac{d\vec{\ell}}{dt} \times \vec{B} = \frac{dQ}{dt}\,d\vec{\ell} \times \vec{B}$$
$$\boxed{= I\,d\vec{\ell} \times \vec{B}}$$

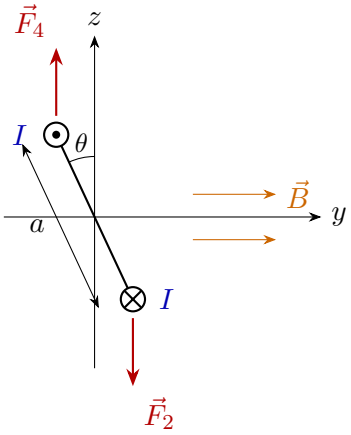
For a constant field $\vec{B}$,

$$\boxed{\vec{F}_m = \int_{\text{wire}} d\vec{F}_m = I\left(\int_{\text{wire}} d\vec{\ell}\right) \times \vec{B} = I\,\vec{\ell} \times \vec{B}}$$

Example: current loop (motor)

A rectangular loop carrying current I can rotate about the x-axis in a uniform field $\vec{B} = B\,\hat{\jmath}$. The sides have lengths a (sides 1 and 3) and b (sides 2 and 4). Tilted by an angle θ from the z-axis (right panel: the loop seen along the x-axis), the forces are:





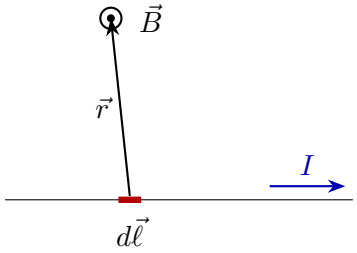
$$\boxed{\vec{F}_1 = I\,a\,B\cos\theta\,\hat{\imath} = -\vec{F}_3} \qquad \text{(cancel)}$$

$$\boxed{\vec{F}_2 = I\,b\,B\,(-\hat{k}) = -\vec{F}_4}$$

and these rotate the coil back to the z-axis.

3.2 Magnetic sources

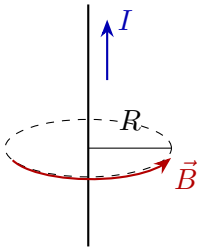
Moving charges source $\vec{B}$-fields.



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2} \tag{3.2}$$

This is the **Biot–Savart law**, where μ_0 is the permeability of free space. Compare with Coulomb's law and ε_0.

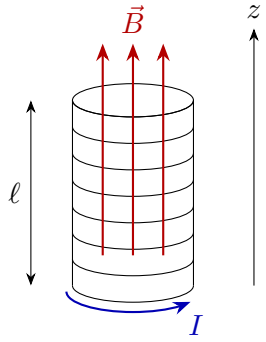
Long wire



$$B = \frac{\mu_0}{2\pi} \frac{I}{R}$$

The direction is given by the RHR. (This requires doing the integral of the Biot–Savart law along the wire.)

Solenoid

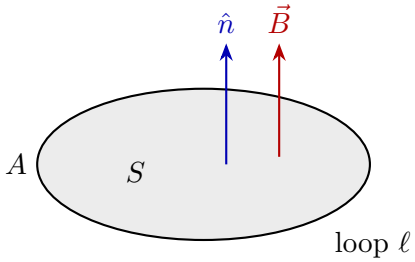


Inside a solenoid of length ℓ with N loops,

$$\vec{B} = \frac{\mu_0 N I}{\ell} \hat{k} \tag{3.3}$$

3.3 Faraday's law and inductance

Faraday's law: changing $\vec{B}$ sources $\vec{E}$.



The **magnetic flux** through a surface S (area A, unit normal $\hat{n}$) bounded by a loop ℓ is

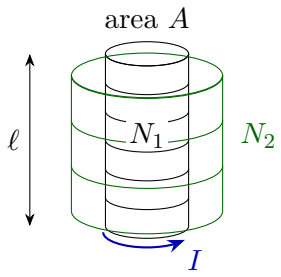
$$\Phi_m = \vec{A} \cdot \vec{B}$$

when $\hat{n} \parallel \vec{B}$ and both are constant. The flux Φ_m is roughly the number of field lines "cut" by the surface.

The voltage around the loop ℓ is

$$\mathcal{E} = -\frac{d\Phi_m}{dt} \tag{3.4}$$

Double solenoid



A current I through an inner solenoid (N_1 turns, length ℓ, area A) induces a voltage in an outer wire wound with N_2 turns. By (3.4) and (3.3),

$$\begin{aligned}\mathcal{E} &= -N_2\frac{d\Phi_m}{dt} = -N_2\frac{d}{dt}(AB)\\ &= -N_2 A\,\frac{d}{dt}\left(\frac{\mu_0 N_1 I}{\ell}\right)\end{aligned}$$

$$\mathcal{E} = -\underbrace{\mu_0\frac{N_1 N_2 A}{\ell}}_{\boxed{M}}\frac{dI}{dt} \tag{3.5}$$

The coefficient M is the **mutual inductance**.

Self-inductance

Changing magnetic fields inside the solenoid induce a voltage that resists the changing current. For a single solenoid, replace $N_2 = N_1 = N$:

$$\mathcal{E} = -\underbrace{\mu_0\frac{N^2 A}{\ell}}_{L}\frac{dI}{dt},$$

where L is the **inductance**. In general,

$$\boxed{\mathcal{E}_L = -L\frac{dI}{dt}} \tag{3.6}$$

Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax University Physics Vol. 2, Chapter 11: Magnetic Forces and Fields

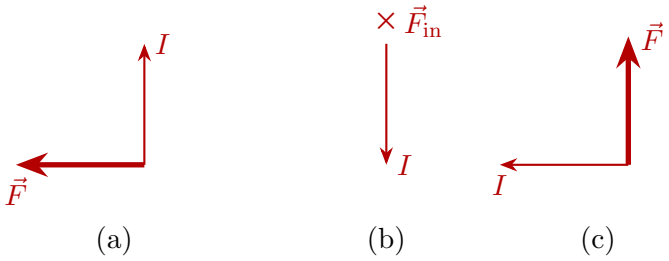
11.21 (a) Aircraft sometimes acquire small static charges. Suppose a supersonic jet has a 0.500 µC charge and flies horizontally at a speed of 660 m/s over Earth's south magnetic pole (near the north geographic pole), where the 8.00×10^{-5} T magnetic field points straight down into the ground. What are the direction and the magnitude of the magnetic force on the plane? (b) Discuss whether the value obtained in part (a) implies this is a significant or negligible effect.

[Answer →]

11.27 (a) An oxygen-16 ion with a mass of 2.66×10^{-26} kg travels at 5.0×10^{6} m/s perpendicular to a 1.20 T magnetic field, which makes it move in a circular arc with a 0.231 m radius. What positive charge is on the ion? (b) What is the ratio of this charge to the charge of an electron? (c) Discuss why the ratio found in (b) should be an integer.

[Answer →]

11.35 What is the direction of the magnetic field that produces the magnetic force shown on the currents in each of the three cases, assuming $\vec{B}$ is perpendicular to I?



(a) (b) (c)

[Answer →]

11.37 (a) A dc power line for a light-rail system carries 1000 A at an angle of 30.0° to Earth's 5.0×10^{-5} T field. What is the force on a 100 m section of this line? (b) Discuss practical concerns this presents, if any.

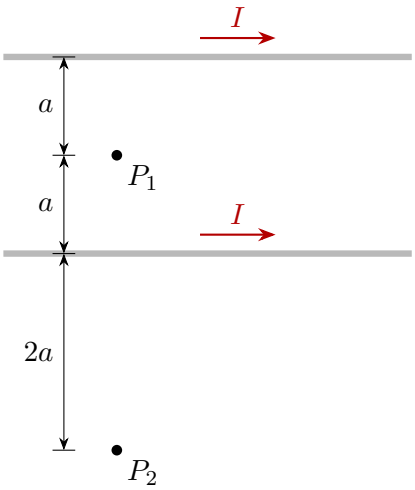
[Answer →]

11.53 A physicist is designing a cyclotron to accelerate protons to one-tenth the speed of light. The magnetic field will have a strength of 1.5 T. Determine (a) the rotational period of the circulating protons and (b) the maximum radius of the protons' orbit.

[Answer →]

OpenStax University Physics Vol. 2, Chapter 12: Sources of Magnetic Fields

12.27 The accompanying figure shows two long, straight, horizontal wires that are parallel and a distance $2a$ apart. If both wires carry current I in the same direction, (a) what is the magnetic field at P_1? (b) P_2?



[Answer →]

12.28 Repeat the calculations of the preceding problem with the direction of the current in the lower wire reversed.

[Answer →]

12.55 A solenoid with 25 turns per centimeter carries a current I. An electron moves within the solenoid in a circle that has a radius of 2.0 cm and is perpendicular to the axis of the solenoid. If the speed of the electron is 2.0×10^5 m/s, what is I?

[Answer →]

OpenStax University Physics Vol. 2, Chapter 14: Inductance

14.31 A coil of 40 turns is wrapped around a long solenoid of cross-sectional area $7.5 \times 10^{-3}\,\text{m}^2$. The solenoid is 0.50 m long and has 500 turns. (a) What is the mutual inductance of this

system? (b) The outer coil is replaced by a coil of 40 turns whose radius is three times that of the solenoid. What is the mutual inductance of this configuration?

[Answer →]

14.39 A coil with a self-inductance of 2.0 H carries a current that varies with time according to $I(t) = (2.0\,\text{A})\sin 120\pi t$. Find an expression for the emf induced in the coil.

[Answer →]

14.43 A long, cylindrical solenoid with 100 turns per centimeter has a radius of 1.5 cm. (a) Neglecting end effects, what is the self-inductance per unit length of the solenoid? (b) If the current through the solenoid changes at the rate 5.0 A/s, what is the emf induced per unit length?

[Answer →]

Chapter 4

Oscillations

This chapter has three objectives:

1. understand the damped oscillator;
2. learn to solve linear, homogeneous differential equations;
3. use trigonometric identities to rearrange formulas.

4.1 Recall: the RC circuit

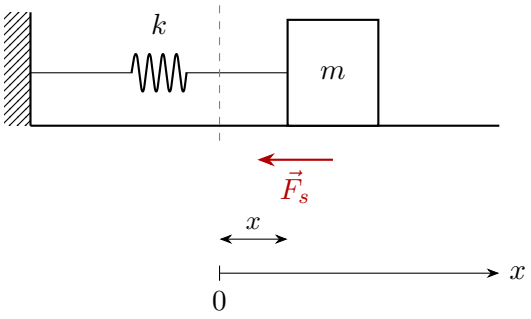
For the discharging capacitor, (2.3) reads (with a dot for d/dt)

$$\dot{Q}(t) = -\frac{1}{RC}\,Q(t).$$

Guess an exponential, $Q(t) = Q_0\,e^{\lambda t}$. Substituting gives

$$\boxed{\lambda = -\frac{1}{RC}} \qquad \therefore \qquad \boxed{Q(t) = Q_0\,e^{-t/RC}}$$

4.2 Simple harmonic oscillator



A mass m on a spring of spring constant k. The spring exerts the force

$$\boxed{F_s = -k\,x}$$

(**Hooke's law**), where x is the displacement from the equilibrium position.

Equation of motion

Newton's second law, $F = ma$ with $F = F_s$, gives $ma = -kx$, that is

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}\,x = 0} \tag{4.1}$$

This is a second-order differential equation.

Solution

Use the same strategy as for the RC circuit. **Guess** $x(t) = A\,e^{\lambda t}$, so

$$\frac{dx}{dt} = \lambda A\,e^{\lambda t}, \qquad \frac{d^2x}{dt^2} = \lambda^2 A\,e^{\lambda t}.$$

Plugging into (4.1) gives $\lambda^2 x + \dfrac{k}{m}\,x = 0$, so

$$\boxed{\lambda = \pm\sqrt{-\frac{k}{m}}}$$

There are two solutions (two derivatives), and $\sqrt{-1} = i$ is imaginary! Call $\lambda = i\omega$, with

$$\omega = \sqrt{\frac{k}{m}}.$$

Then

$$\boxed{x(t) = A\,e^{+i\omega t} + B\,e^{-i\omega t}} \tag{4.2}$$

which we call **Form 1**. How do we interpret it?

Euler's formula

Use Euler's formula

$$\boxed{e^{i\omega t} = \cos\omega t + i\sin\omega t}$$

Then

$$\begin{aligned} x(t) &= A\,(\cos\omega t + i\sin\omega t) + B\,(\cos\omega t - i\sin\omega t) \\ &= \underbrace{(A+B)}_{\equiv C}\cos\omega t + \underbrace{i\,(A-B)}_{\equiv D}\sin\omega t \end{aligned}$$

$$\boxed{x(t) = C\cos\omega t + D\sin\omega t} \tag{4.3}$$

which is **Form 2**. Here ω is the *angular frequency* of the oscillations, and C, D are set by the initial conditions. Why? Because

$$\boxed{x(t=0) = C} \qquad v(t) = -C\omega\sin\omega t + D\omega\cos\omega t \implies \boxed{v(t=0) = D\,\omega}$$

Frequency and period

$$\omega = 2\pi f, \qquad T = \frac{1}{f} = \frac{2\pi}{\omega},$$

where f is the *frequency* of the oscillations and T the *period*.

One more identity

Use $\cos(a+b) = \cos a\cos b - \sin a\sin b$:

$$E\cos(\omega t + \phi) = \underbrace{(E\cos\phi)}_{\equiv C}\cos\omega t - \underbrace{(E\sin\phi)}_{\equiv D}\sin\omega t.$$

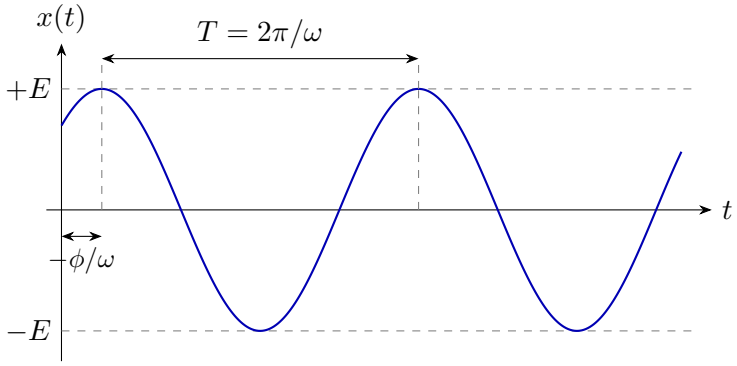
Note that

$$C^2 + D^2 = E^2\cos^2\phi + E^2\sin^2\phi = E^2 \qquad -\frac{D}{C} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

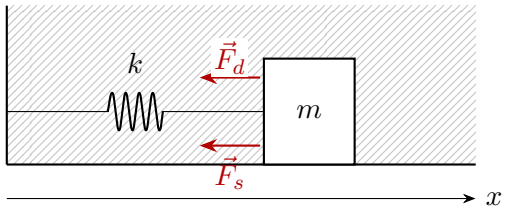
Therefore

$$x(t) = E\cos(\omega t + \phi) \tag{4.4}$$

which is **Form 3**: a shifted oscillating function with amplitude E.



4.3 Damped oscillator



A box in a damping medium (e.g. air). There is now a damping force

$$\vec{F}_d = -b\,\vec{v}$$

where b is the **damping coefficient**.

Newton's second law, $ma = -kx - bv$, gives

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\,x = 0 \tag{4.5}$$

Solution

Guess $x(t) = A\,e^{\lambda t}$. Then (4.5) becomes $m\lambda^2 + b\lambda + k = 0$, so

$$\lambda = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}.$$

Call

$$\tau = \frac{2m}{b}, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Then if $\omega_0^2 > 1/\tau^2$ the square root is imaginary and

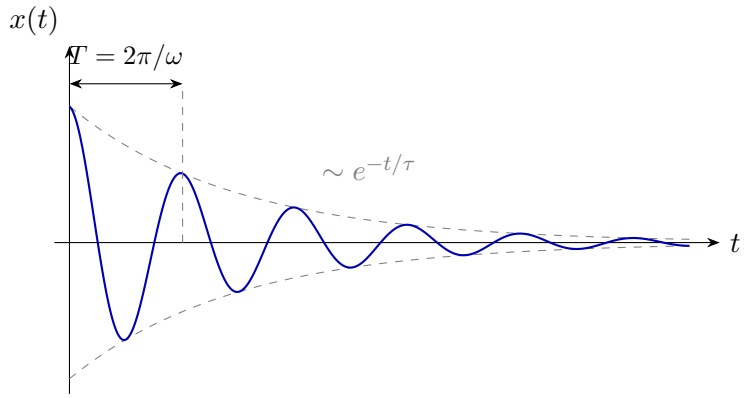
$$x(t) = e^{-t/\tau}\left(A\,e^{i\omega t} + B\,e^{-i\omega t}\right) \qquad \text{(Form 1)}$$

$$= E\,e^{-t/\tau}\cos(\omega t + \phi) \qquad \text{(Form 3)}$$

where

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

This is a **damped oscillator**.



Problems

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OpenStax University Physics Vol. 1, Chapter 15: Oscillations

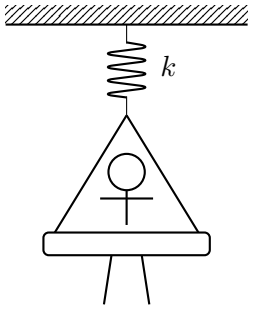
15.29 A mass m_0 is attached to a spring and hung vertically. The mass is raised a short distance in the vertical direction and released. The mass oscillates with a frequency f_0. If the mass is replaced with a mass nine times as large, and the experiment was repeated, what would be the frequency of the oscillations in terms of f_0?

[Answer →]

15.39a A novelty clock has a 0.0100 kg-mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position?

[Answer →]

15.59 The device pictured in the figure entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring. (a) If the spring stretches 0.250 m while supporting an 8.0 kg child, what is its force constant? (b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?



[Answer →]

15.67a A 2.00 kg block lies at rest on a frictionless table. A spring, with a spring constant of 100 N/m is attached to the wall and to the block. A second block of 0.50 kg is placed on top of the first block. The 2.00 kg block is gently pulled to a position $x = +A$ and released from rest. There is a coefficient of friction of 0.45 between the two blocks. What is the period of the oscillations?

[Answer →]

OpenStax Calculus Vol. 3, Chapter 7: Second-Order Differential Equations

7.93a An 800 lb weight (25 slugs) is attached to a vertical spring with a spring constant of 226 lb/ft. The system is immersed in a medium that imparts a damping force equal to 10 times the

instantaneous velocity of the mass. Find the equation of motion if it is released from a position 20 ft below its equilibrium position with a downward velocity of 41 ft/s.

(*Note:* take $m = 800\,\text{lb}/g$, where $g = 32\,\text{ft}\,\text{s}^{-2}$.)

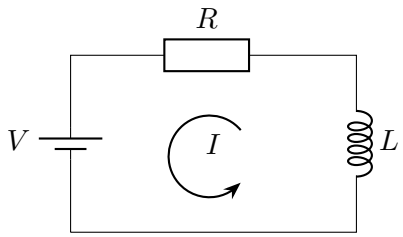
[Answer →]

Chapter 5

RL, RLC and AC Circuits

5.1 RL circuits

Recall that a changing current through an inductor induces a voltage that opposes the change, $\mathcal{E}_L = -L\,dI/dt$ (3.6), where the **inductance** L depends only on geometry.



Consider a voltage source V, a resistor R and an inductor L in series. Because of the definition of inductance, the inductor contributes $\mathcal{E}_L = -L\,dI/dt$ to the loop rule:

$$V - IR - L\frac{dI}{dt} = 0 \tag{5.1}$$

This is the same equation as for the RC circuit, with $Q \leftrightarrow I$. Recall (2.2), the loop rule $V - IR - Q/C = 0$ with $I = dQ/dt$:

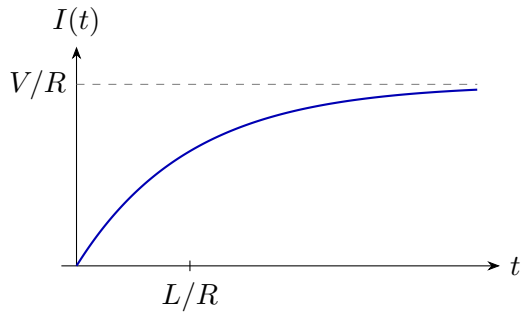
$$V - \frac{Q}{C} - R\frac{dQ}{dt} = 0, \qquad Q(t) = VC\left(1 - e^{-t/RC}\right).$$

Under the replacements

$$Q \to I, \qquad R \to L, \qquad C = \frac{1}{R}$$

this becomes (5.1), so

$$I(t) = \frac{V}{R}\left(1 - e^{-(R/L)\,t}\right)$$



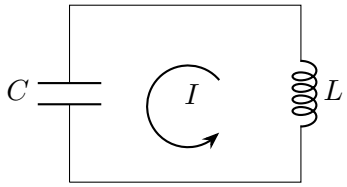
The time constant is L/R (compare RC). The inductor opposes changes in the current: I does not jump to V/R but grows gradually.

Connection with the oscillations chapter

We can also solve (5.1) with the strategy of the Oscillations chapter. The homogeneous equation $L\,dI/dt + RI = 0$ is solved by guessing $I = I_0\,e^{\lambda t}$, which gives $\lambda = -R/L$. A constant $I = V/R$

solves the full equation, so $I(t) = V/R + I_0\, e^{-(R/L)t}$, and $I(0) = 0$ fixes $I_0 = -V/R$: the same result. This split into a *homogeneous* and a *particular* solution reappears for the driven RLC circuit below.

5.2 LC circuits



With no source, the loop rule is $-L\, dI/dt - Q/C = 0$. Using $I = dQ/dt$,

$$\boxed{\frac{d^2Q}{dt^2} = -\frac{Q}{LC}} \tag{5.2}$$

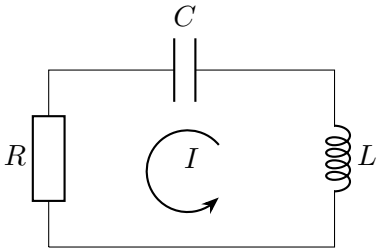
This is a harmonic oscillator!

Compare (4.1): Q plays the role of x and $1/LC$ the role of k/m. By (4.4),

$$\boxed{Q(t) = Q_0 \cos(\omega t + \phi)}\,, \qquad \boxed{\omega = \frac{1}{\sqrt{LC}}}$$

where ω is the **natural angular frequency** of the circuit.

5.3 RLC circuits



Adding a resistor to the LC circuit, the loop rule becomes

$$\boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0} \tag{5.3}$$

This is a damped oscillator!

Connection with the damped oscillator

Recall (4.5), $m\,\ddot{x} + b\,\dot{x} + kx = 0$. Equation (5.3) is the same equation with

$$m = 1, \qquad b = \frac{R}{L}, \qquad k = \frac{1}{LC}, \qquad x \leftrightarrow Q.$$

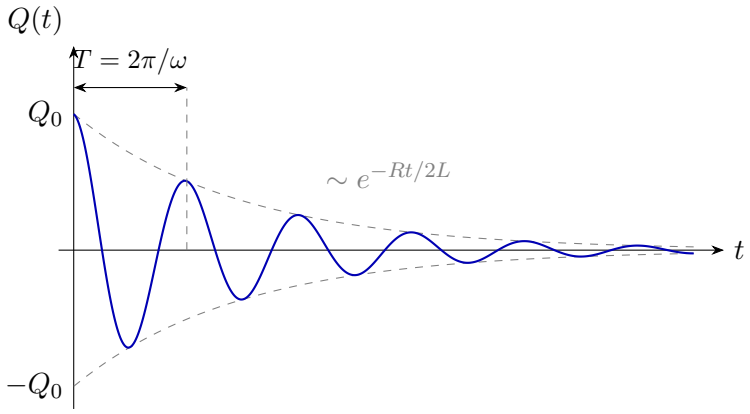
(Multiplying (5.3) by L gives the equivalent dictionary: inductance $\leftrightarrow$ mass, resistance $\leftrightarrow$ damping coefficient b, $1/C \leftrightarrow$ spring constant k.) The decay time $\tau = 2m/b$ becomes $2L/R$, and $\omega_0 = \sqrt{k/m}$ becomes the LC frequency $1/\sqrt{LC}$.

The solution of the damped oscillator then gives, in the underdamped case $1/LC > (R/2L)^2$,

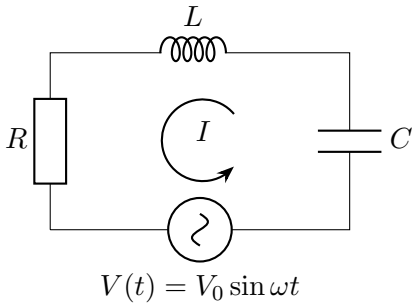
$$\boxed{Q(t) = Q_0\, e^{-(R/2L)\,t} \cos(\omega t + \phi)}$$

with

$$\boxed{\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}\,, \qquad T = \frac{2\pi}{\omega}.$$



5.4 Driven RLC circuit



Now add a source $V(t) = V_0 \sin \omega t$. The loop rule (K2) gives

$$V_0 \sin \omega t - L \frac{d^2Q}{dt^2} - R \frac{dQ}{dt} - \frac{Q}{C} = 0 \tag{5.4}$$

Strategy: homogeneous plus particular solution

Solve for $Q = Q_h + Q_p$ (the general solution). The homogeneous part Q_h solves the equation without the source, $L\ddot{Q}_h + R\dot{Q}_h + Q_h/C = 0$, so by the previous section

$$Q_h(t) = Q_0\, e^{-(R/2L)\,t} \cos(\omega' t + \phi), \qquad \omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

This decays exponentially, so at late times only the particular solution Q_p remains.

Finding Q_p

Guess a response at the driving frequency,

$$Q_p = \frac{A}{\omega} \cos \omega t + \frac{B}{\omega} \sin \omega t.$$

Then the current we are looking for is

$$I_p = \frac{dQ_p}{dt} = -A \sin \omega t + B \cos \omega t, \qquad \frac{d^2Q_p}{dt^2} = -A\omega \cos \omega t - \omega B \sin \omega t.$$

Plugging into (5.4) gives

$$\begin{aligned} V_0 \sin \omega t = {} & L\,(-A\omega \cos \omega t - B\omega \sin \omega t) \\ & + R\,(-A \sin \omega t + B \cos \omega t) \\ & + \frac{1}{\omega C}\,(A \cos \omega t + B \sin \omega t) \end{aligned}$$

Define

$$R_{LC} := \omega L - \frac{1}{\omega C}.$$

Matching the terms in $\cos\omega t$:

$$-L\omega A + RB + \frac{1}{\omega C}A = 0 \iff -R_{LC}A + RB = 0 \implies \boxed{A = \frac{R}{R_{LC}}B}$$

and in $\sin\omega t$:

$$-L\omega B - RA + \frac{1}{\omega C}B = V_0 \iff -R_{LC}B - RA = V_0 \implies \left(R_{LC} + \frac{R^2}{R_{LC}}\right)B = -V_0.$$

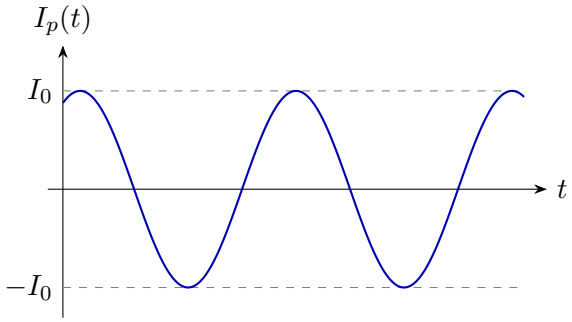
Hence

$$\boxed{B = -V_0\,\frac{R_{LC}}{R_{LC}^2 + R^2}} \qquad \boxed{A = -V_0\,\frac{R}{R_{LC}^2 + R^2}}$$

Amplitude and phase of the current

The particular current $I_p = B\cos\omega t - A\sin\omega t$ has the form $C\cos\omega t + D\sin\omega t$ of the Oscillations chapter, so by (4.4)

$$I_p(t) = I_0\cos(\omega t + \phi), \qquad I_0 = \sqrt{A^2 + B^2}, \qquad \tan\phi = \frac{A}{B}.$$



The current oscillates at the driving frequency ω, with the amplitude I_0 found next.

Substituting A and B,

$$\boxed{I_0 = V_0\sqrt{\frac{R_{LC}^2 + R^2}{\left(R_{LC}^2 + R^2\right)^2}} = \frac{V_0}{\sqrt{R_{LC}^2 + R^2}}}$$

Equivalently $I_0 = V_0/R_{\text{eff}}$ with the effective resistance

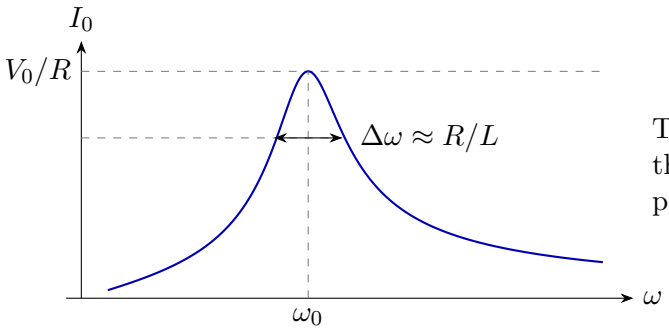
$$R_{\text{eff}} = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}.$$

Resonance

The amplitude I_0 is largest when $R_{LC} = \omega L - 1/(\omega C) = 0$, that is at

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

the natural frequency of the LC circuit, where $I_0 = V_0/R$. This is **resonance**!



The width of the resonance peak is $\Delta\omega \approx R/L$: the smaller the resistance, the sharper the peak.

Quality factor

The **quality factor** Q (not to be confused with the charge Q) measures how narrow the resonance is,

$$\boxed{\mathsf{Q} = \frac{\omega_0}{\Delta\omega}}$$

where $\Delta\omega$ is the width of the peak between the points where the dissipated power ($\propto I_0^2$) has dropped to half its maximum, i.e. where $I_0 = I_{\max}/\sqrt{2}$, equivalently $|R_{LC}| = R$. When the peak is narrow (small R), we can expand around ω_0:

$$R_{LC} = \frac{L\left(\omega^2 - \omega_0^2\right)}{\omega} \approx 2L\left(\omega - \omega_0\right),$$

so $|R_{LC}| = R$ at $|\omega - \omega_0| = R/2L$, a full width $\Delta\omega \approx R/L$. Therefore

$$\boxed{\mathsf{Q} \approx \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

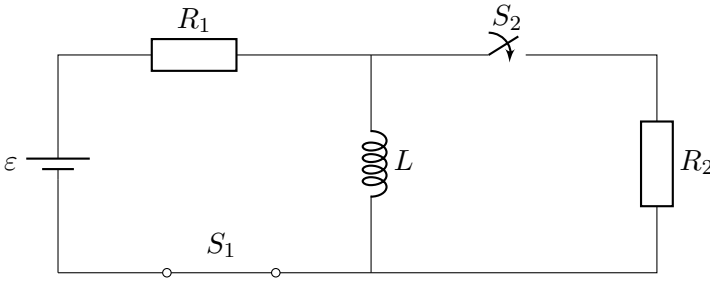
Using $\tau = 2L/R$ from the damped oscillator, this is also $\mathsf{Q} \approx \omega_0\tau/2$: a large Q means weak damping and a sharp resonance.

Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax University Physics Vol. 2, Chapter 14: Inductance

14.53 For the circuit shown, $\varepsilon = 20\,\text{V}$, $L = 4.0\,\text{mH}$, and $R = 5.0\,\Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter (at $t = 0$) S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4}\,\text{s}$, and (c) the voltages across L and R_2 at $t = 4.0 \times 10^{-4}\,\text{s}$. $R_1 = R_2 = R$.



[Answer →]

14.65 In an oscillating LC circuit, the maximum charge on the capacitor is $2.0 \times 10^{-6}\,\text{C}$ and the maximum current through the inductor is 8.0 mA. (a) What is the period of the oscillations? (b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

[Answer →]

14.71 In an oscillating RLC circuit with $L = 10\,\text{mH}$, $C = 1.5\,\mu\text{F}$, and $R = 2.0\,\Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

[Answer →]

OpenStax University Physics Vol. 2, Chapter 15: Alternating-Current Circuits

15.39 For an RLC series circuit, $R = 100\,\Omega$, $L = 150\,\text{mH}$, and $C = 0.25\,\mu\text{F}$. (a) If an ac source of variable frequency is connected to the circuit, at what frequency is maximum power dissipated in the resistor? (b) What is the quality factor of the circuit?

[Answer →]

15.41 (a) What is the resonant frequency of a resistor, capacitor, and inductor connected in series if $R = 100\,\Omega$, $L = 2.0\,\text{H}$, and $C = 5.0\,\mu\text{F}$? (c) What is the Q of the circuit? (d) What is the bandwidth of the circuit?

[Answer →]

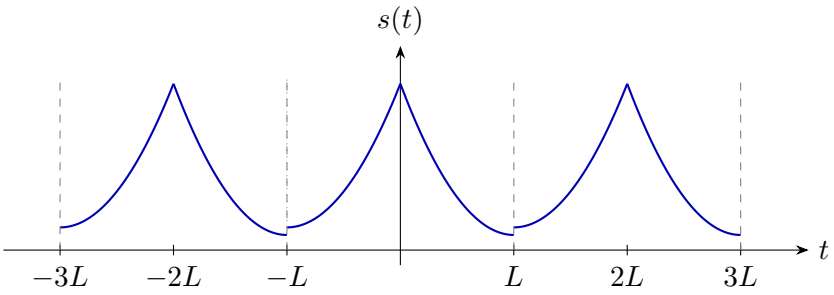
Chapter 6

Fourier Series

6.1 Fourier series

Consider a periodic function $s(t)$ that repeats outside the interval $t \in [-L, L]$ and satisfies:

1. $\int_{-L}^{L} |s(t)|\, dt < \infty$,
2. $s(t)$ has a finite number of discontinuities,
3. $s(t)$ has a finite number of extrema.



Note: $s(t)$ could also just be defined only on $[-L, L]$.

Then it is possible to write $s(t)$ as an infinite sum of sine and cosine functions:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{\pi n t}{L}\right) + b_n \sin\left(\frac{\pi n t}{L}\right) \right] \tag{6.1}$$

Idea: the coefficients (a_n, b_n) are like the "components" of an infinite-dimensional "vector", where the orthonormal "basis" elements are sines and cosines. For example, a vector in three dimensions is $\vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath} + v_z\,\hat{k}$:

$$(v_x, v_y, v_z) \;\to\; (a_n, b_n), \qquad (\hat{\imath}, \hat{\jmath}, \hat{k}) \;\to\; (\sin, \cos).$$

Orthogonality

Here the inner product is now an integral over $[-L, L]$. For vectors, $\langle \vec{v}, \vec{u} \rangle = \sum_i v_i u_i$; for functions

$$\langle s(t), r(t) \rangle = \frac{1}{L} \int_{-L}^{L} s(t)\, r(t)\, dt\,.$$

For the unit vectors, $\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1$ and $\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{k}=\hat{\imath}\cdot\hat{k}=0$. The sines and cosines satisfy the same relations, for integers $m,n\geq 1$:

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\sin\left(\frac{m\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)=\delta_{mn}} \tag{6.2}$$

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\cos\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)=\delta_{mn}} \tag{6.3}$$

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\sin\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)=0} \tag{6.4}$$

Check. Use $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$. For $m\neq n$ both cosines integrate to zero over a whole number of periods; for $m=n$ the first term is 1 and the second integrates to zero, so the integral is $\frac{1}{L}\cdot\frac{1}{2}\cdot 2L=1$. The cosine–cosine case is identical with $\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$, and the sine–cosine integrand is odd, so it integrates to zero over the symmetric interval.

The coefficients

Multiply (6.1) by $\cos(m\pi t/L)$ and integrate. For $m\neq 0$, using (6.3) and (6.4):

$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)\cos\left(\frac{m\pi t}{L}\right)=\frac{1}{L}\int_{-L}^{L}dt\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right]\cos\left(\frac{m\pi t}{L}\right)=\sum_{n=1}^{\infty}(a_n\delta_{mn}+0)=a_m\,.$$

For $m=0$:

$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)=\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\,dt=\frac{a_0}{2}\cdot\frac{2L}{L}=a_0\,.$$

Similarly, using (6.2) and (6.4),

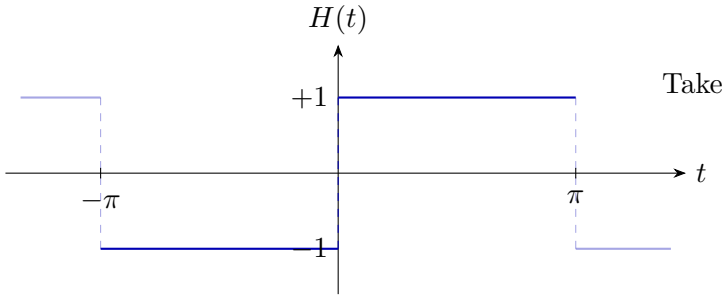
$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)\sin\left(\frac{m\pi t}{L}\right)=0+\sum_{n=1}^{\infty}(0+b_n\delta_{mn})=b_m\,.$$

Collecting these results:

$$\boxed{a_n=\frac{1}{L}\int_{-L}^{L}dt\,s(t)\cos\left(\frac{n\pi t}{L}\right),\quad n\geq 0} \tag{6.5}$$

$$\boxed{b_n=\frac{1}{L}\int_{-L}^{L}dt\,s(t)\sin\left(\frac{n\pi t}{L}\right),\quad n\geq 1} \tag{6.6}$$

6.2 Example: the step function



Take $L=\pi$ and $s(t)=H(t)$, where

$$H(t)=\begin{cases}-1, & -\pi<t<0,\\ +1, & 0<t<\pi.\end{cases}$$

Since H is odd and $\cos(nt)$ is even, the integrand of a_n is odd and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, H(t)\cos(nt) = 0\,.$$

For b_n the integrand is even (odd $\times$ odd), so we can double the integral over $[0,\pi]$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, H(t)\sin(nt) = \frac{2}{\pi}\int_0^{\pi} dt\, \sin(nt) = -\frac{2}{\pi}\,\frac{\cos(nt)}{n}\Big|_0^{\pi} = \frac{2}{n\pi}\,(1-(-1)^n)\,,$$

using $\cos(n\pi) = +1$ for even n and -1 for odd n, and $\cos 0 = 1$. Hence

$$b_n = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd},\\[2mm] 0, & n \text{ even}, \end{cases} \qquad \text{so} \qquad H(t) = \frac{4}{\pi}\left[\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right].$$

6.3 From Fourier series to the Fourier transform

The Fourier series (6.1) describes a function that repeats with period $2L$. A signal that does not repeat, such as a single pulse, can be thought of as the limit $L\to\infty$ of a periodic one. Let us see what happens to the series in this limit.

Complex form

Write $\omega_n = n\pi/L$ for the frequency of the nth term. Then (6.1), (6.5) and (6.6) read

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right], \qquad \begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\cos(\omega_n t'),\\ b_n &= \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\sin(\omega_n t'). \end{aligned}$$

The neighbouring frequencies are separated by $\Delta\omega = \omega_{n+1}-\omega_n = \pi/L$. Combine the two coefficients into one complex number,

$$F_n \equiv a_n - i\,b_n = \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\left[\cos(\omega_n t') - i\sin(\omega_n t')\right] = \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\, e^{-i\omega_n t'},$$

where we used Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Since $(a_n - ib_n)\,e^{i\omega_n t} = a_n\cos(\omega_n t) + b_n\sin(\omega_n t) + i\,[a_n\sin(\omega_n t) - b_n\cos(\omega_n t)]$, each term of the series is the real part of $F_n e^{i\omega_n t}$:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\mathrm{Re}\big[F_n\, e^{i\omega_n t}\big]\,.$$

The limit $L\to\infty$

As L grows, the coefficients F_n shrink like $1/L$, so we define the rescaled quantity $F(\omega_n)\equiv L\,F_n$, which stays finite. Using $1/L = \Delta\omega/\pi$ the series becomes

$$s(t) = \frac{a_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\mathrm{Re}\big[F(\omega_n)\, e^{i\omega_n t}\big]\,\Delta\omega\,, \qquad F(\omega_n) = \int_{-L}^{L} dt'\, s(t')\, e^{-i\omega_n t'}.$$

When $L\to\infty$:

- the spacing $\Delta\omega = \pi/L \to 0$, so the allowed frequencies $\omega_n = n\Delta\omega$ fill out a continuous variable ω and the sum over n becomes an integral over $\omega\in(0,\infty)$;

- $a_0/2 = \frac{1}{2L}\int s\,dt \to 0$, because $\int |s|\,dt$ is finite.

We therefore obtain the **Fourier transform** of $s(t)$,

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} dt\, s(t)\, e^{-i\omega t}} \tag{6.7}$$

and the representation of the signal

$$s(t) = \frac{1}{\pi}\,\mathrm{Re}\int_0^{\infty} d\omega\, F(\omega)\, e^{i\omega t}.$$

The inverse transform

Since $s(t)$ is real, complex conjugating (6.7) gives $F(\omega)^\star = F(-\omega)$. The complex conjugate of the integrand $F(\omega)e^{i\omega t}$ is therefore $F(-\omega)e^{-i\omega t}$, and $\mathrm{Re}\, z = \frac{1}{2}(z + z^\star)$ gives

$$\mathrm{Re}\int_0^{\infty} d\omega\, F(\omega)e^{i\omega t} = \frac{1}{2}\int_0^{\infty} d\omega\, F(\omega)e^{i\omega t} + \frac{1}{2}\int_0^{\infty} d\omega\, F(-\omega)e^{-i\omega t} = \frac{1}{2}\int_{-\infty}^{\infty} d\omega\, F(\omega)e^{i\omega t},$$

where the second term was rewritten by substituting $\omega \to -\omega$. Hence

$$\boxed{s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{i\omega t}} \tag{6.8}$$

Compare this with the Fourier series: the discrete frequencies ω_n and coefficients (a_n, b_n) have been replaced by a continuous frequency ω and the complex function $F(\omega)$, whose real part plays the role of a_n and minus its imaginary part that of b_n (up to the overall factor L). The signal conditions are the same as for the series, with $\int_{-\infty}^{\infty} |s(t)|\,dt < \infty$.

Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

Fourier problem set

These problems are from the course (not from the sources above).

1 (a) Plot the waveform of the periodic function defined as

$$f(t) = -\frac{2t}{\pi} \quad \text{if} \quad -\pi \le t \le \pi, \qquad f(t) = f(t + 2\pi) \quad \text{for all } t\,.$$

(b) Compute the Fourier series of this function for all n.

[Answer →]

2 (a) Using the provided python code `fourier_sawtooth.py`, plot the 'sawtooth' waveform as a function of time for a value of w not equal to zero, one or 1/2. Try to pick a value that will highlight interesting features of the Fourier series and explain why you made your choice. What is special about the values $w = 0$, $w = 1/2$ and $w = 1$?

(b) For the value of w chosen above, plot the Fourier coefficients for up to $N = 15$.

(c) Choose 3 values of N (different from 15) for which to plot the N^{th} order Fourier approximation that illustrate different characteristics of the approximation. Explain what is happening as N gets large and why you chose the values you did.

[Answer →]

3 Modify `fourier_sawtooth.py` to check your answer to problem 1. Make a plot to compare the analytic solution obtained to the numerical computation (by plotting the difference, plotting them side-by-side, or some other way).

[Answer →]

Chapter 7

Introduction to Signal Processing

7.1 Introduction

We live in the information age. Every moment of every day we are bombarded with information in the form of text, audio, images and video. Most of this information is processed and transmitted digitally. This means that, after being generated, the information must be encoded into some sort of digital format. This information is then manipulated, transmitted, decoded, and reproduced in a form that we can receive. In the first part of this course, we have focused on the electromagnetic components that make the generation, transmission and reproduction of such information possible. Now we will focus on the processes involved in encoding, manipulating and decoding this information. These processes are collectively known as *signal processing* because they convert information into the form of a signal and then process its content.

Perhaps the most useful aspect of modern signal processing is that it provides methods for dramatically compressing information. This involves manipulating a signal in such a way that the compressed media is perceptibly *indistinguishable* from the original. Good video compression can reduce the size of a file by a factor of around 100 without any perceptible loss of quality. This extraordinary savings in storage has made video streaming on smart phones possible. Imagine how cumbersome it would be if every video you watched on social media took 100 times longer to load! The tools of signal processing are what enable such a dramatic increase in performance. We will now learn the basics of these techniques.

Representing signals

The subject of study in signal processing is, perhaps unsurprisingly, a *signal*. At the most basic level, a signal is a sequence of values that change in time. This is most easily represented as a (possibly continuous) sequence of numbers. Mathematically, this is nicely written as a real function $f(t)$ that is everywhere bounded, has a finite number of maxima and discontinuities, and has a beginning and end. We have seen earlier in this course (Fourier Series) that these are exactly the requirements for the signal to have a Fourier transform. From now on, we will assume that a Fourier transform of our signals exists.

When a signal is represented as a function of time, we say that it is in the *time domain*. When it is represented in Fourier space, we say that it is in the *frequency domain*, and will call it $F(\omega)$. It is often convenient to use a Fourier transform in complex exponential form so that $F(\omega)$ is a complex number. But since $f(t)$ is real, $F(\omega)$ is constrained to obey the following identity:[1]

$$F(\omega)^\star = F(-\omega)\,, \tag{7.1}$$

[1] Note that this condition follows trivially from taking the complex conjugate of the definition, (6.7), of the Fourier transform.

where $\star$ is the complex conjugate operator. Here the Fourier transform is $F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,\mathrm{d}t$, which was derived from the Fourier series in the chapter on Fourier series, Equation (6.7). Note that since $f(t)$ has a start, t_s, and a finish, t_f, this integral needs to only be performed over (t_s, t_f) in practice.

Let us now explore the utility of the Fourier transform $F(\omega)$. One particularly noteworthy fact is that almost all signals used in media applications are much more efficient to encode when represented in Fourier space. Take the example of an audio signal, which will form the basis of most of our considerations in what follows. Many sounds, including the human voice and (non-percussive) musical instruments, are produced by resonating phenomena.

Think of a piano string that can only vibrate in specific frequencies that are integer multiples of a fundamental frequency, which is inversely proportional to the length of the string.

All of this can easily be understood in Fourier space. The different harmonics will show up as a series of different peaks centred on definite frequencies with different amplitudes. The strongest, and usually the lowest, of these will represent the note we identify as being played, which is called the *fundamental* frequency of the sound. In addition, each peak will have some characteristic shape that adds a distinctive character to the sound. In the bottom plot of Figure 7.1, we show the spectrum of a note played on a piano. Each of the harmonics appears as a sharp peak centred on equally spaced frequencies. For the note being played in the figure, the fundamental is at 440 Hz, which is the "A4" note on a piano. The fundamental and its harmonics are marked by dashed grey lines in the plot. We can see that there are sharp peaks of decreasing intensity at exactly the frequencies we'd expect. We see from this example that the Fourier transform of an audio signal, especially that of a nearly pure tone like a piano, has very distinctive features. This illustrates that the Fourier transform can paint a relatively simple picture of the structure of a sound that reflects the physical processes involved in creating that sound.

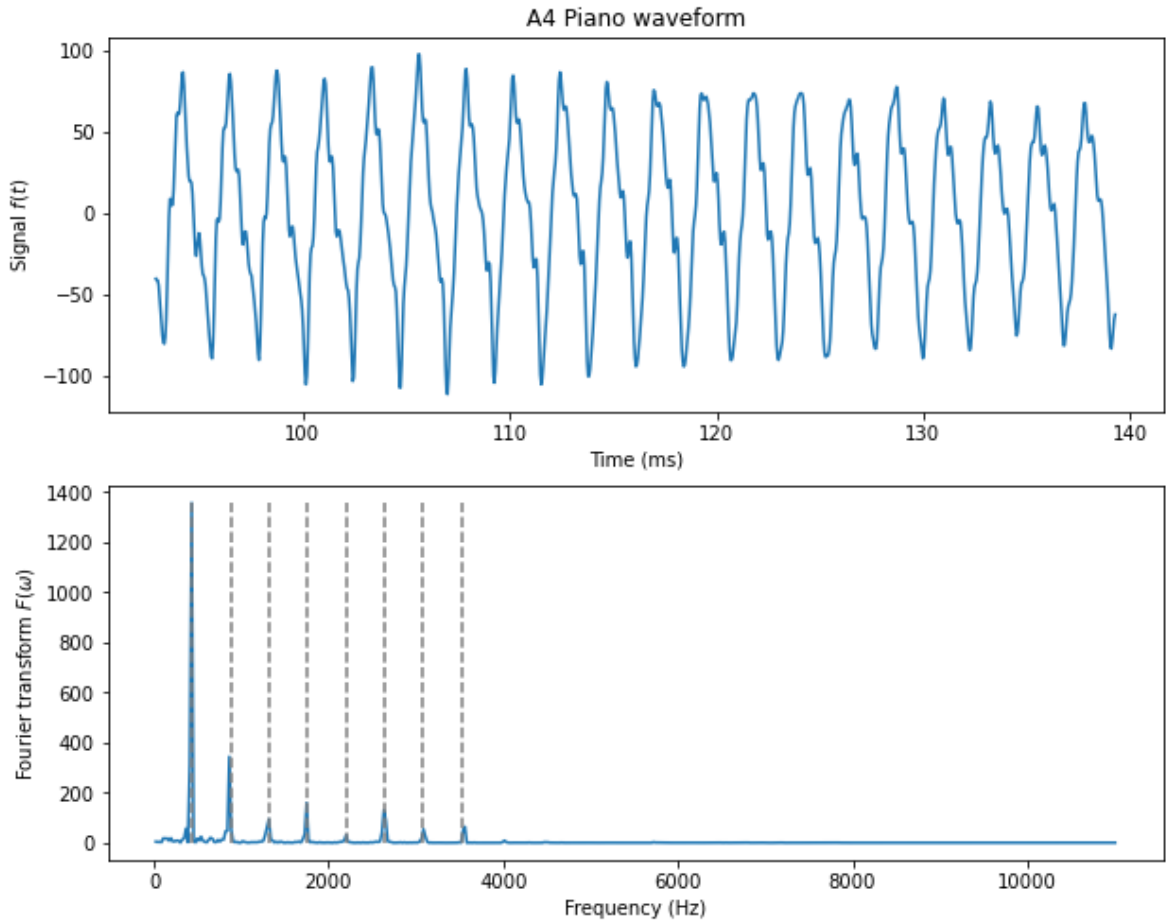


Figure 7.1: **Top:** The time domain signal of a piano playing the note A4, which has a frequency of 440 Hz. Note that this waveform has some bumpy features and longer modulations that are not straightforward to interpret.
Bottom: The frequency domain representation of the same note. Notice the sharp peaks occurring at regular 440 Hz intervals. The first eight are indicated by the dashed lines.

Compare this to the original representation of the audio signal of the piano in the time domain. Here we clearly see that the signal is oscillating but that each oscillation is characterised by a waveform with some rather unpredictable features that are difficult to interpret. On a fine-grained scale, the signal looks bumpy, and it's a bit difficult to predict exactly where these bumps will occur. On a coarse-grained level, the oscillations seem to be modulating as the amplitudes of the oscillations change unpredictably. This not only makes the signal more difficult to understand, it also makes it much less efficient to encode digitally. We will see now why.

7.2 The basics of signal compression

Entropy coding

The signal at the top of Figure 7.1 has been encoded digitally. This means that it is represented by a series of numbers, called *samples*, recorded at even intervals of time (in this case 22050 samples per second). The value of the amplitude of each sample must be recorded with some finite degree of accuracy. In this case, the amplitude is stored as an integer in the range $(-128, 128)$. That means the digitisation of the signal has introduced an error of roughly 1 part in 256 for each sample taken. To reduce this error, you can store each sample in terms of a larger range of numbers. But doing so will require more data storage per sample. The amount of bits of data required to store

each sample determines the *bitrate* of the signal, which is just the total number of bits required to store the signal per second. One important goal of signal processing is to reduce the bitrate of a signal without affecting the quality of the signal.

The simplest way to do this is to notice that signals that don't change very rapidly require less storage space per sample. Consider the Fourier transform at the bottom of Figure 7.1. Most of the spectrum is almost completely flat. Above 4000 Hz, the Fourier transformed signal stays below a value of 8. For all of these frequencies it would be much more efficient to store each sampled frequency using a number in the range $(0, 7)$. You could thus store exactly the same signal at a much smaller bitrate.

The process of storing signals by adjusting the bitrate to the flatness of the signal is called *entropy coding*. Clearly *no* information is lost in such a coding because you are still storing the same values for each sample. This is called *lossless compression* because no information is lost in the compression process. However, the amount of compression that is possible depends entirely on the properties of the signal. For the flat Fourier transform at the bottom of Figure 7.1, entropy coding gives an enormous savings of storage since the signal itself is mostly flat. Contrastingly, the original signal is constantly changing between nearly the full range of values $(-128, 128)$. This means that most samples will have to be stored with nearly the full bitrate for each sample. Thus, entropy coding will provide almost no additional savings of storage in this case.

The amount of compression provided by entropy coding can be roughly quantified by a function of the signal called its *spectral flatness*.[2] The less spectrally flat a signal, the less information is required to store the signal, and therefore the more compression you get from entropy coding. One of the main benefits of the Fourier transform is that it is usually much more spiky, and therefore less spectrally flat, than the original signal. This means that, by taking the Fourier transform of a signal, you get bigger compression effects when entropy coding.

Perceptual models for audio signals: the threshold in quiet

To get more significant compression one can eliminate aspects of the signal that don't affect its overall quality. This is called *lossy compression* because information is lost in the signal. The secret to lossy compression is to know what aspects of the signal can be thrown away without affecting the signal's purpose. This, in turn, requires an understanding of the signal and how it's being used.

As an example, let us consider audio signals. In this case, the signal represents a sound wave that is fed into a speaker to be heard by a human. Any sounds that cannot be heard by the human ear can be removed from the signal without any loss of quality.

The challenge, then, is to determine what sounds cannot be heard by the (typical) human ear. In general, this requires a perceptual model of the human auditory system. In the simplest model, all sounds that are too soft to be heard are removed. To build such a model, all we have to do is measure how loud a particular frequency needs to be before a typical human can just barely detect it. Frequencies outside a certain range, for example those of a dog whistle, cannot be detected no matter how loud they are. But inside this range, there is a threshold that can be measured below which a pure tone of any frequency cannot be heard. This threshold is called the human *threshold in quiet*. It is measured at a particular frequency with no other sounds present to drown it out. The threshold in quiet is measured in decibels (dB) and is approximated by the following function:

$$T(f) = Af^{-4/5} - Be^{-3/5(f-f_0)^2} + Cf^4\,, \tag{7.2}$$

[2]Formally, the spectral flatness is defined as the ratio of the geometric and arithmetic means $= \frac{(\prod x_n)^{1/N}}{\frac{1}{N}\sum x_n}$, where x_n is the n^{th} sample of the signal and N is the total number of samples. It has a maximum at 1, which is approached for white noise, and a minimum at zero.

where $A = 3.64$, $B = 6.5$, $C = 10^{-3}$, $f_0 = 3.3$ and the frequency f is in kHz. This function is shown in Figure 7.2.

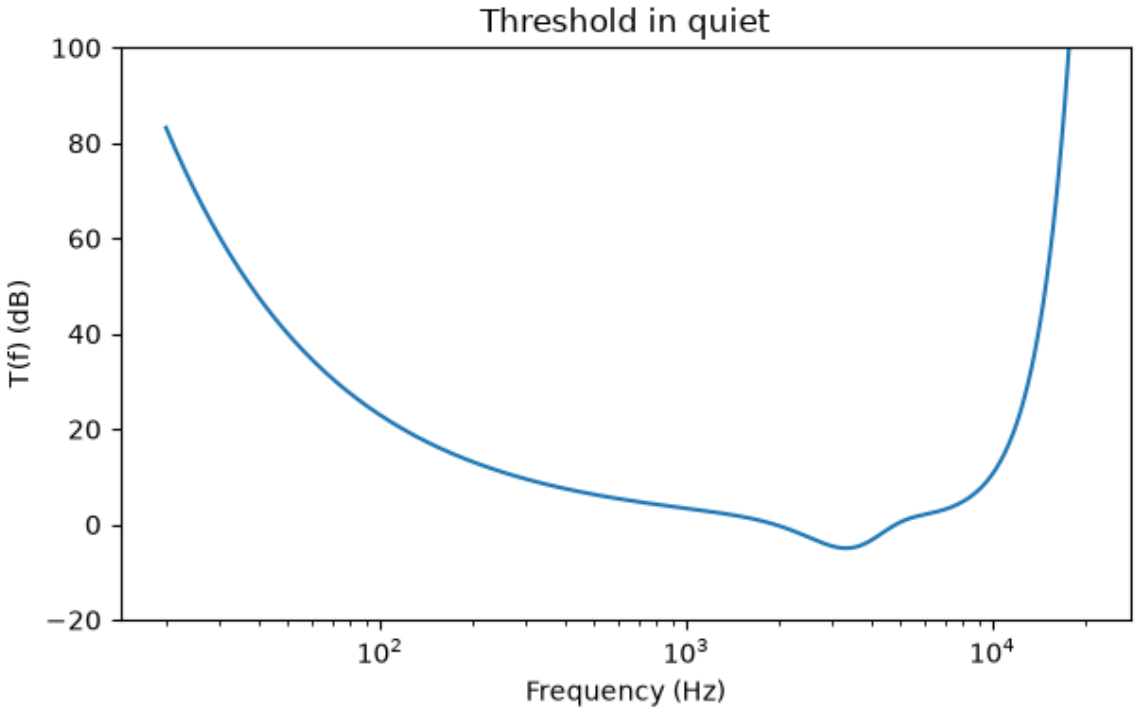


Figure 7.2: The threshold in quiet, Equation (7.2). The formula is from Bosi and Goldberg (2003) (see Sources and Attribution).

To compress a signal, one can simply compare the amplitude of the Fourier transform converted into dB with those of the threshold $T(f)$ and throw away any that fall below it. This will not lead to any perceptual loss of quality because these signals are imperceptible to the human ear.[3] By throwing away the frequencies that can't be heard, we compress the signal by reducing the number of frequencies to store in memory.

This simple algorithm can be applied to the signal shown in Figure 7.1. Figure 7.3 shows the signal (top) and its Fourier transform (bottom) converted into decibels following the prescription described in Footnote 3. The orange line is the function $T(f)$ of (7.2) representing the threshold in quiet. Only 11% of the frequencies in this signal rise above $T(f)$. For such a simple signal, this results in a significant amount of compression: roughly a factor of 9 compression.[4]

[3] In practice, converting a Fourier transform into decibels, dB, is done by taking $\text{dB} = 20 \log_{10} A$, where A is the amplitude of the Fourier transform of a pure sine wave with amplitude equal to the smallest signal that can be recorded at the given bitrate (eg, the amplitude of this smallest sine wave is 1 for the signal of Figure 7.1). This somewhat arbitrary definition is suitable for most purposes as long as the gain of the amplified audio output produces a sound with a comfortable listening volume.

[4] Note that the signal at the frequencies that do remain is large and will typically require more memory than each sample of the original signal so that the exact compression factor is tricky to calculate.

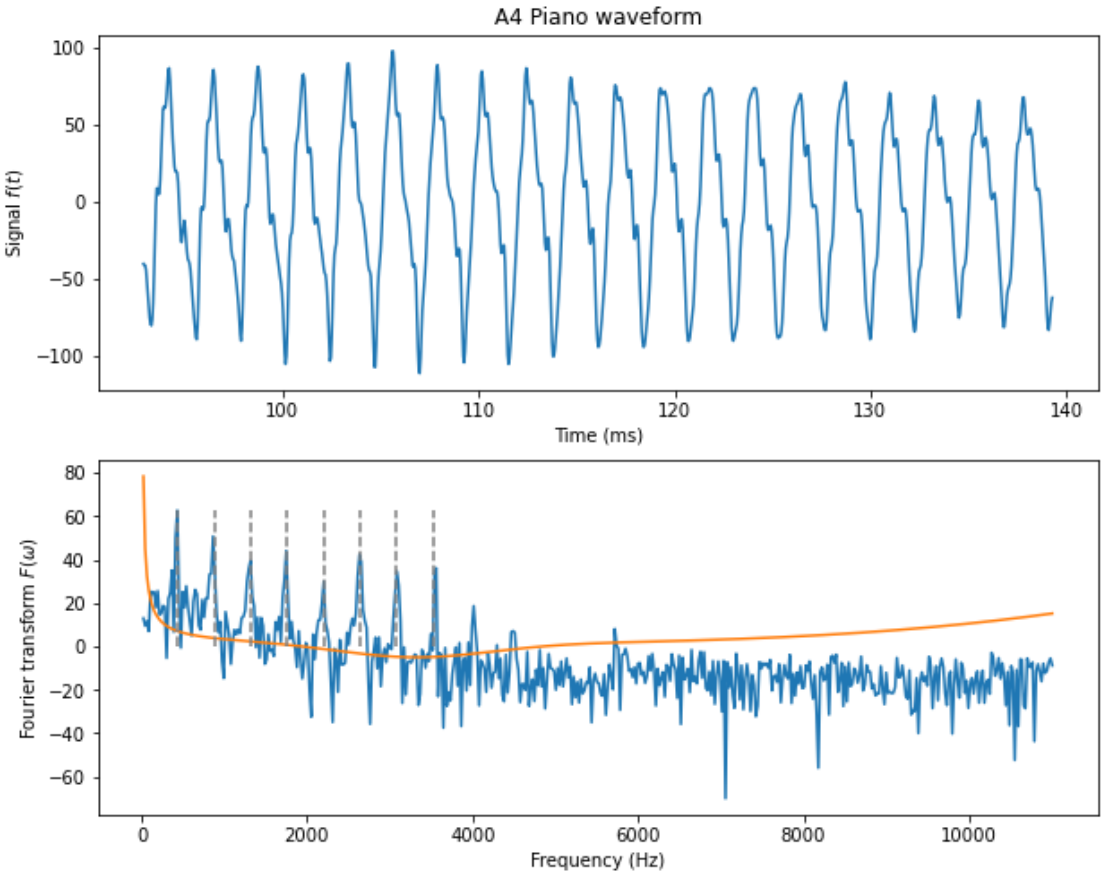


Figure 7.3: The signal of Figure 7.1 (top) with its Fourier transform converted into dB (bottom). The first eight harmonics are marked by grey dashed lines. The orange line is the threshold in quiet, $T(f)$ or Equation (7.2). 89% of frequencies in this signal fall below $T(f)$.

7.3 Filters

The compression algorithm described in the section on the threshold in quiet requires the ability to perform a Fourier transform on some input signal. For many applications, like audio and video streaming, the encoding and decoding must be done in real time. While this is fast enough to do on a modern computer equipped with the right coding libraries — especially after the development of the so-called *Fast Fourier Transform (FFT)* algorithm — this hasn't always been the case. In the past, and even for some modern applications such as hearing aids, it is therefore useful to have other methods for isolating the Fourier components of a signal.

What we require is an electronic device that can output a signal whose Fourier transform has effectively been multiplied by a function that drops smoothly to zero in a certain frequency range. A device that achieves this is called a *filter* because it "filters" out frequencies that lie in a specified range.

There are three different kinds of filters that correspond to the kinds of frequencies the filter cuts out. A filter that removes high frequencies is called a *low-pass* filter because only frequencies below a certain threshold, ω_0, are allowed to "pass" through. Conversely, a *high-pass* filter removes low frequencies and allows frequencies above the threshold, ω_0, to pass. Finally a *band-pass* filter only allows frequencies that fall into a certain range $(\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2)$ to pass. In this notation, ω_0 is in the middle of the frequency "band" and $\Delta\omega$ is the width of the band.

Mathematically, each of the filters above has the effect of multiplying the Fourier transform of a signal $F(\omega)$ by some function $h(\omega)$ so that $F(\omega) \to h(\omega)F(\omega)$. The function $h(\omega)$ is itself the

Fourier transform as the so-called *linear response function* of a filter. We can therefore characterise a filter by $h(\omega)$. This function might have different properties for different kinds of filters but the universal features that distinguish low, high and band pass filters are the following:

$$\begin{aligned} h(\omega) &\approx 1 \text{ for } \omega \ll \omega_0 \\ &\approx 0 \text{ for } \omega \gg \omega_0 \end{aligned} \qquad \text{(low-pass)} \tag{7.3}$$

$$\begin{aligned} h(\omega) &\approx 0 \text{ for } \omega \ll \omega_0 \\ &\approx 1 \text{ for } \omega \gg \omega_0 \end{aligned} \qquad \text{(high-pass)} \tag{7.4}$$

$$\begin{aligned} h(\omega) &\approx 1 \text{ for } \omega_0 - \Delta\omega/2 < \omega < \omega_0 + \Delta\omega/2 \\ &\approx 0 \text{ for } \omega \gg \omega_0 \\ &\approx 0 \text{ for } \omega \ll \omega_0 \end{aligned} \qquad \text{(band-pass)} \tag{7.5}$$

We'll see now that we can build filters with response functions like this using the circuits we've studied in this course.

Low-pass filter

First consider the simple RC-circuit shown in Figure 7.4 below. The idea is to apply the signal $s(t) \equiv v_{\text{in}}$ as a changing voltage onto the left side of the circuit. We now want to consider the output voltage v_{out} that will be generated across the capacitor. Using the definition of capacitance, we can relate v_{out} to the charge on the capacitor: $v_{\text{out}} = Q/C$. Similarly, using Ohm's law, we can relate the voltage across the resistor to the current $I = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}v_{\text{out}}}{\mathrm{d}t}$, which is just proportional to the time derivative of the output voltage.

$$RC\frac{\mathrm{d}v_{\text{out}}}{\mathrm{d}t} + v_{\text{out}} = v_{\text{in}}\,. \tag{7.6}$$

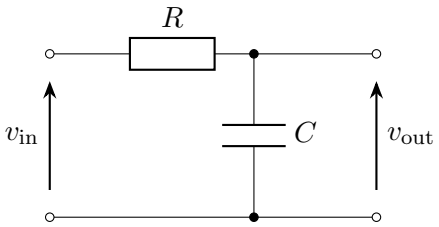


Figure 7.4: An RC circuit set up as a low-pass filter.

The Fourier transform is a very useful tool for solving this differential equation as long as we assume that the signal v_{in} drops to zero outside of some time interval (t_s, t_f). Let us rewrite the input and output signals using the (inverse) Fourier transforms $\tilde{v}_{\text{in}}$ and $\tilde{v}_{\text{out}}$ defined as

$$\begin{aligned} v_{\text{in}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{v}_{\text{in}}\mathrm{d}\omega\,, \\ v_{\text{out}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{v}_{\text{out}}\mathrm{d}\omega\,. \end{aligned} \tag{7.7}$$

When this is inserted into (7.6) the following algebraic equation is obtained:

$$\tilde{v}_{\text{out}} = \left(\frac{1}{1 + i\frac{\omega}{\omega_0}}\right)\tilde{v}_{\text{in}}\,, \tag{7.8}$$

where $\omega_0 = \frac{1}{RC}$ is the cut-off frequency of the filter.

To understand why the circuit in Figure 7.4 is a low-pass filter, we can note that the circuit has the effect of multiplying the Fourier transform of the input signal by the function

$$h(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}} \,. \tag{7.9}$$

The amplitude of this function is:

$$|h(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}} \,. \tag{7.10}$$

This circuit therefore behaves like a low-pass filter because for $\omega \ll \omega_0$, $|h(\omega)| \to 1$ and for $\omega \gg \omega_0$, $|h(\omega)| \to 0$ as required by (7.3). Note that $|h|$ does not switch abruptly from 1 to 0: the transition is spread over a range of frequencies comparable to ω_0 itself, so this simple circuit is only a rough approximation to an ideal low-pass filter. Sharper cutoffs require more complicated circuits (see the last subsection below).

High-pass filter

We can create a high-pass filter using the RC-circuit shown in Figure 7.5. Kirchhoff's second law for that circuit gives: $Q/C + v_{\text{out}} = v_{\text{in}}$. Taking time derivatives of this equation and using the fact that $\frac{\mathrm{d}Q}{\mathrm{d}t} = v_{\text{out}}/R$ we obtain the following differential equation for v_{out}:

$$\frac{\mathrm{d}v_{\text{out}}}{\mathrm{d}t} + \frac{v_{\text{out}}}{RC} = \frac{\mathrm{d}v_{\text{in}}}{\mathrm{d}t} \,. \tag{7.11}$$

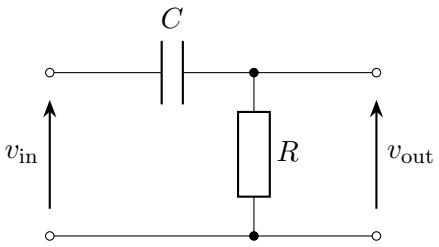


Figure 7.5: An RC circuit set up as a high-pass filter.

Using the (inverse) Fourier transform definition (7.7), this leads to the following Fourier transform of the linear response function

$$h(\omega) = \frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} = \frac{i\omega/\omega_0}{1 + i\omega/\omega_0} \,, \tag{7.12}$$

which has an amplitude

$$|h(\omega)| = \frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_0)^2}} \,. \tag{7.13}$$

This amplitude goes to zero when $\omega \ll \omega_0$ and one when $\omega \gg \omega_0$. The circuit thus behaves like a high-pass filter according to our requirement (7.4).

Band-pass filters

We can construct a band-pass filter using the RLC-circuit of Figure 7.6. In this case, the output voltage is measured across the resistor. Kirchhoff's second law gives

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} + v_{\text{out}} = v_{\text{in}} \,. \tag{7.14}$$

Differentiating once and multiplying by R/L gives the following differential equation for v_{out}:

$$\frac{\mathrm{d}^2 v_{\text{out}}}{\mathrm{d}t^2} + \Delta\omega \frac{\mathrm{d}v_{\text{out}}}{\mathrm{d}t} + \omega_0^2 v_{\text{out}} = \Delta\omega \frac{\mathrm{d}v_{\text{in}}}{\mathrm{d}t}\,, \tag{7.15}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}}\,, \qquad \Delta\omega = \frac{R}{L}\,.$$

Using the inverse Fourier transforms as before leads to the following response function on Fourier space

$$h(\omega) = \frac{i\omega\Delta\omega}{(\omega_0^2 - \omega^2) + i\omega\Delta\omega}\,, \tag{7.16}$$

which has amplitude

$$|h(\omega)| = \frac{1}{\sqrt{\left(\frac{\omega_0^2 - \omega^2}{\omega\Delta\omega}\right)^2 + 1}}\,. \tag{7.17}$$

Provided we choose R, L, and C to be such that $\Delta\omega < \omega_0$, then this amplitude will go to zero when $\omega \gg \omega_0$ and $\omega \ll \omega_0$. When $\omega = \omega_0$, $|h(\omega)| = 1$. This means that the response function allows frequencies within a band of rough size $\Delta\omega$ around ω_0 and stops frequencies outside that band. This is exactly what one would expect for a band-pass filter according to our requirement (7.5).

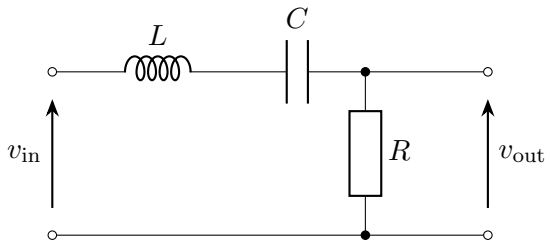


Figure 7.6: An RLC circuit set up as a band-pass filter.

Note that this has the same interpretation as the resonance of the driven RLC circuit, discussed in the section on resonance of the chapter on RL, RLC and AC circuits, and as Equation (15.15) of the OpenStax *University Physics Volume 2* online book used in this course. The width $\Delta\omega = R/L$ is the same width as there, and the quality factor of the circuit is $\mathsf{Q} = \omega_0/\Delta\omega$. This is because, if we adjust the circuit such that $\Delta\omega \ll \omega_0$, this frequency response curve will be highly peaked on ω_0. This means that the circuit is tuned to, or *resonating* at, the frequency ω_0.

Signal processing with filters

Using various combinations of low, high and band pass filters, it is possible to isolate and eliminate frequencies in the same way this is done directly with a Fourier transform. Note also that more complicated versions of these circuits can be used to provide cleaner realisations of the functions satisfying (7.3), (7.4) and (7.5). It is even possible to add amplification to the signal, as one might do in a radio receiver. Thus, any compression technique based on a perceptual model of the human auditory system can be implemented using electronic circuits of the kind discussed here. Ultimately, this is because the human auditory system itself is effectively a system of mechanical and electrical filters that performs its own processing on the sounds waves entering the ear. The goal of compression algorithms is to mimic this processing and eliminate any information that is unnecessary.

Early compression algorithms, like MPEG 1 and 2,[5] run the signal through an array of 32 band-pass filters centred on evenly spaced frequency intervals. This array of filters is called a *filter-bank*. These different bands are supposed to mimic the different absorption bands found in the inner ear. One can then analyse the signal in each of these bands and determine which ones are either below the threshold in quiet or are drowned out by louder sounds in neighbouring bands.[6] The bands that are found to be too soft to be heard can then either be eliminated or stored at a much lower band-width than the original signal. During playback, the 32 bands are recombined to produce a sound that is imperceptibly different from the original but that is much more spectrally flat and, therefore, takes much less memory to store. More modern formats like MP4, use Fast Fourier Transform algorithms to analyse and manipulate the signal directly in Fourier space. This requires faster computing power that is not always available for smaller devices like hearing aids. The underlying mathematical principles, however, are the same.

[5] For example, files with the 'MP3' file extensions

[6] This is a phenomenon called *masking*.

Answers

Final results only. Come to class to ask how.

Chapter 1: Electric Fields & Capacitance

5.53 $F = 53.94\,\text{N}$ [← Problem]

5.63 $\vec{F}_{\text{net}} = \left[k\frac{q^2}{a^2} + k\frac{q^2}{2a^2}\frac{1}{\sqrt{2}}\right]\hat{\imath} - \left[k\frac{q^2}{a^2} + k\frac{q^2}{2a^2}\frac{1}{\sqrt{2}}\right]\hat{\jmath}$ [← Problem]

5.95 $\theta = 32.0°$ [← Problem]

8.27 $500\,\mu\text{C}$ [← Problem]

8.33 $0.29\,\mu\text{F}$ [← Problem]

8.37 $11.4\,\mu\text{F}$ [← Problem]

Chapter 2: Current, Resistance & DC Circuits

9.27 $I(t) = -I_{\max}\sin(\omega t + \phi)$ [← Problem]

9.37 $R = 0.10\,\Omega$ [← Problem]

9.51 $R_{\min} = 2.375 \times 10^5\,\Omega,\ I_{\min} = 12.63\,\mu\text{A};\quad R_{\max} = 2.625 \times 10^5\,\Omega,\ I_{\max} = 11.43\,\mu\text{A}$ [← Problem]

10.31 (a) 0.74 A; (b) 0.742 A [← Problem]

10.39 $V_1 = 42\,\text{V},\ V_2 = 6\,\text{V},\ R_4 = 6\,\Omega$ [← Problem]

10.43 $I_1 = \frac{2}{3}\frac{V}{R},\ I_2 = \frac{V}{3R},\ I_3 = \frac{V}{3R}$ [← Problem]

10.53 (a) 12.3 mA; (b) 7.50×10^{-4} s; (c) 4.53 mA; (d) 3.89 V [← Problem]

10.57 $R = 3.33 \times 10^{-3}\,\Omega$ [← Problem]

Chapter 3: Magnetic Fields and Inductors

† Not in the OpenStax answer key; derived for these notes.

11.21 (a) 2.64×10^{-8} N, north; (b) the force is very small, so the effect is negligible [← Problem]

11.27 (a) 4.80×10^{-19} C; (b) 3; (c) charge comes in integer multiples of the electron charge, with no free charges smaller than this [← Problem]

11.35 (a) into the page; (b) left; (c) out of the page [← Problem]

11.37 (a) 2.50 N; (b) the power lines must be attached so that they are not moved by the force from Earth's magnetic field [← Problem]

11.53 (a) $4.4 \times 10^{-8}\,\mathrm{s}$; (b) $0.21\,\mathrm{m}$ [← Problem]

12.27 At P_1 the net field is zero; at P_2, $B = \dfrac{3\mu_0 I}{8\pi a}$ into the page [← Problem]

12.28 At P_1, $B = \dfrac{\mu_0 I}{\pi a}$ into the page; at P_2, $B = \dfrac{\mu_0 I}{8\pi a}$ out of the page† [← Problem]

12.55 $I = 0.0181\,\mathrm{A}$ [← Problem]

14.31 (a) $3.8 \times 10^{-4}\,\mathrm{H}$; (b) $3.8 \times 10^{-4}\,\mathrm{H}$ [← Problem]

14.39 $\varepsilon = 480\pi \sin(120\pi t - \pi/2)\,\mathrm{V}$ [← Problem]

14.43 (a) $0.089\,\mathrm{H/m}$; (b) $0.44\,\mathrm{V/m}$ [← Problem]

Chapter 4: Oscillations

15.29 $f = \dfrac{1}{3} f_0$ [← Problem]

15.39a $0.335\,\mathrm{m/s}$ [← Problem]

15.59 (a) $314\,\mathrm{N/m}$; (b) $1.00\,\mathrm{s}$; (c) $1.25\,\mathrm{m/s}$ [← Problem]

15.67a $0.99\,\mathrm{s}$ [← Problem]

7.93a $x(t) = e^{-t/5}\,(20\cos 3t + 15\sin 3t)$ [← Problem]

Chapter 5: RL, RLC and AC Circuits

14.53 (a) $4.0\,\mathrm{A}$; (b) $2.4\,\mathrm{A}$; (c) on R_2: $12\,\mathrm{V}$; on L: $12\,\mathrm{V}$ [← Problem]

14.65 (a) $1.6 \times 10^{-3}\,\mathrm{s}$; (b) $3.9 \times 10^{-4}\,\mathrm{s}$ [← Problem]

14.71 $6.9\,\mathrm{ms}$ [← Problem]

15.39 (a) $820\,\mathrm{Hz}$; (b) 7.8 [← Problem]

15.41 (a) $50\,\mathrm{Hz}$; (c) 6.32; (d) $50\,\mathrm{rad\,s^{-1}}$ [← Problem]

Chapter 6: Fourier Series

1 (b) $a_n = 0$ for all n; $b_n = \dfrac{4(-1)^n}{n\pi}$ for $n \geq 1$ [← Problem]

2 (a) $w = 0$ and $w = 1$ give a falling and a rising linear ramp; $w = 1/2$ gives a symmetric triangle wave [← Problem]

3 The numerical b_n agree with $4(-1)^n/(n\pi)$ and the numerical a_n vanish (up to numerical error) [← Problem]