Chapter 4

Oscillations

This chapter has three objectives:

1. understand the damped oscillator;
2. learn to solve linear, homogeneous differential equations;
3. use trigonometric identities to rearrange formulas.

4.1 Recall: the RC circuit

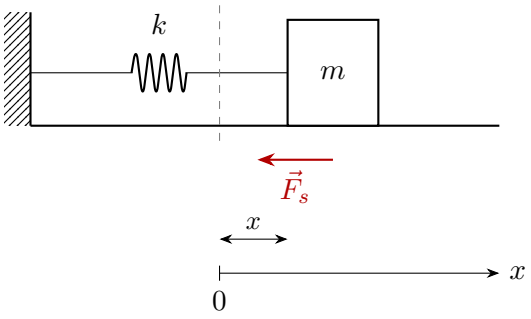
For the discharging capacitor, (2.3) reads (with a dot for d/dt)

$$\dot{Q}(t) = -\frac{1}{RC}\,Q(t).$$

Guess an exponential, $Q(t) = Q_0\,e^{\lambda t}$. Substituting gives

$$\boxed{\lambda = -\frac{1}{RC}} \qquad \therefore \qquad \boxed{Q(t) = Q_0\,e^{-t/RC}}$$

4.2 Simple harmonic oscillator



A mass m on a spring of spring constant k. The spring exerts the force

$$\boxed{F_s = -k\,x}$$

(**Hooke's law**), where x is the displacement from the equilibrium position.

Equation of motion

Newton's second law, $F = ma$ with $F = F_s$, gives $ma = -kx$, that is

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}\,x = 0} \tag{4.1}$$

This is a second-order differential equation.

Solution

Use the same strategy as for the RC circuit. **Guess** $x(t) = A\,e^{\lambda t}$, so

$$\frac{dx}{dt} = \lambda A\,e^{\lambda t}, \qquad \frac{d^2x}{dt^2} = \lambda^2 A\,e^{\lambda t}.$$

Plugging into (4.1) gives $\lambda^2 x + \dfrac{k}{m}\,x = 0$, so

$$\boxed{\lambda = \pm\sqrt{-\frac{k}{m}}}$$

There are two solutions (two derivatives), and $\sqrt{-1} = i$ is imaginary! Call $\lambda = i\omega$, with

$$\omega = \sqrt{\frac{k}{m}}.$$

Then

$$\boxed{x(t) = A\,e^{+i\omega t} + B\,e^{-i\omega t}} \tag{4.2}$$

which we call **Form 1**. How do we interpret it?

Euler's formula

Use Euler's formula

$$\boxed{e^{i\omega t} = \cos\omega t + i\sin\omega t}$$

Then

$$\begin{aligned} x(t) &= A\,(\cos\omega t + i\sin\omega t) + B\,(\cos\omega t - i\sin\omega t) \\ &= \underbrace{(A+B)}_{\equiv C}\cos\omega t + \underbrace{i\,(A-B)}_{\equiv D}\sin\omega t \end{aligned}$$

$$\boxed{x(t) = C\cos\omega t + D\sin\omega t} \tag{4.3}$$

which is **Form 2**. Here ω is the *angular frequency* of the oscillations, and C, D are set by the initial conditions. Why? Because

$$\boxed{x(t=0) = C} \qquad v(t) = -C\omega\sin\omega t + D\omega\cos\omega t \implies \boxed{v(t=0) = D\,\omega}$$

Frequency and period

$$\omega = 2\pi f, \qquad T = \frac{1}{f} = \frac{2\pi}{\omega},$$

where f is the *frequency* of the oscillations and T the *period*.

One more identity

Use $\cos(a+b) = \cos a\cos b - \sin a\sin b$:

$$E\cos(\omega t + \phi) = \underbrace{(E\cos\phi)}_{\equiv C}\cos\omega t - \underbrace{(E\sin\phi)}_{\equiv D}\sin\omega t.$$

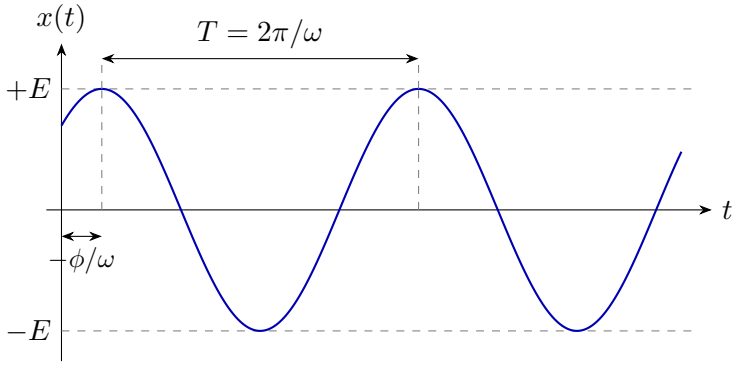
Note that

$$C^2 + D^2 = E^2\cos^2\phi + E^2\sin^2\phi = E^2 \qquad -\frac{D}{C} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

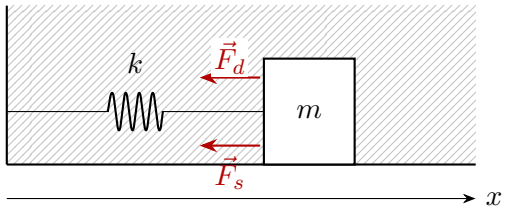
Therefore

$$x(t) = E\cos(\omega t + \phi) \tag{4.4}$$

which is **Form 3**: a shifted oscillating function with amplitude E.



4.3 Damped oscillator



A box in a damping medium (e.g. air). There is now a damping force

$$\vec{F}_d = -b\,\vec{v}$$

where b is the **damping coefficient**.

Newton's second law, $ma = -kx - bv$, gives

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\,x = 0 \tag{4.5}$$

Solution

Guess $x(t) = A\,e^{\lambda t}$. Then (4.5) becomes $m\lambda^2 + b\lambda + k = 0$, so

$$\lambda = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}.$$

Call

$$\tau = \frac{2m}{b}, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Then if $\omega_0^2 > 1/\tau^2$ the square root is imaginary and

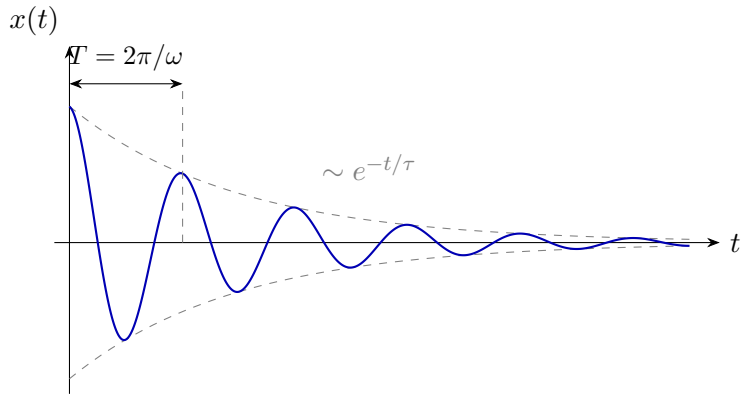
$$x(t) = e^{-t/\tau}\left(A\,e^{i\omega t} + B\,e^{-i\omega t}\right) \qquad \text{(Form 1)}$$

$$= E\,e^{-t/\tau}\cos(\omega t + \phi) \qquad \text{(Form 3)}$$

where

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

This is a **damped oscillator**.



Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax University Physics Vol. 1, Chapter 15: Oscillations

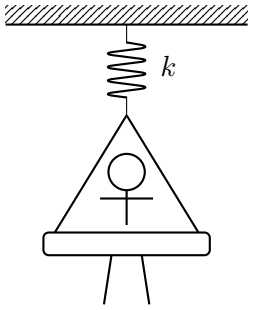
15.29 A mass m_0 is attached to a spring and hung vertically. The mass is raised a short distance in the vertical direction and released. The mass oscillates with a frequency f_0. If the mass is replaced with a mass nine times as large, and the experiment was repeated, what would be the frequency of the oscillations in terms of f_0?

[Answer →]

15.39a A novelty clock has a 0.0100 kg-mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position?

[Answer →]

15.59 The device pictured in the figure entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring. (a) If the spring stretches 0.250 m while supporting an 8.0 kg child, what is its force constant? (b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?



[Answer →]

15.67a A 2.00 kg block lies at rest on a frictionless table. A spring, with a spring constant of 100 N/m is attached to the wall and to the block. A second block of 0.50 kg is placed on top of the first block. The 2.00 kg block is gently pulled to a position $x = +A$ and released from rest. There is a coefficient of friction of 0.45 between the two blocks. What is the period of the oscillations?

[Answer →]

OpenStax Calculus Vol. 3, Chapter 7: Second-Order Differential Equations

7.93a An 800 lb weight (25 slugs) is attached to a vertical spring with a spring constant of 226 lb/ft. The system is immersed in a medium that imparts a damping force equal to 10 times the

instantaneous velocity of the mass. Find the equation of motion if it is released from a position 20 ft below its equilibrium position with a downward velocity of 41 ft/s.

(*Note:* take $m = 800\,\text{lb}/g$, where $g = 32\,\text{ft}\,\text{s}^{-2}$.)

[Answer →]

Answers

Chapter 4: Oscillations

15.29 $f = \frac{1}{3} f_0$ [← Problem]

15.39a $0.335\,\mathrm{m/s}$ [← Problem]

15.59 (a) $314\,\mathrm{N/m}$; (b) $1.00\,\mathrm{s}$; (c) $1.25\,\mathrm{m/s}$ [← Problem]

15.67a $0.99\,\mathrm{s}$ [← Problem]

7.93a $x(t) = e^{-t/5}\,(20\cos 3t + 15\sin 3t)$ [← Problem]

Sources and Attribution

The end-of-chapter problems in these notes are taken from the following openly licensed textbooks. Problem numbers in the text (for example 5.53) are the numbers of the end-of-chapter problems in the book named in the problems section.

- OpenStax, *University Physics Volume 2*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-2
- OpenStax, *University Physics Volume 1*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-1
- OpenStax, *Calculus Volume 3*, by Gilbert Strang, Edwin "Jed" Herman et al. Licensed under CC BY 4.0. https://openstax.org/details/books/calculus-volume-3

The signal-processing chapter also draws on the following book, which is not openly licensed and is cited rather than reproduced: Marina Bosi and Richard E. Goldberg, *Introduction to Digital Audio Coding and Standards*, Kluwer Academic Publishers, 2003. The approximation of the threshold in quiet used there is taken from that book; the plot of it is our own.

The Fourier problems at the end of the Fourier chapter were written for this course by the author.

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