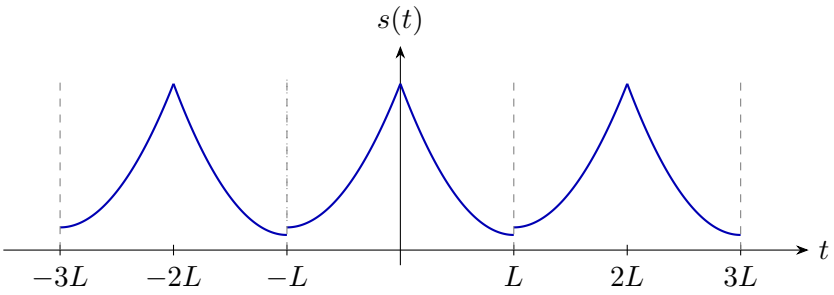
Chapter 6

Fourier Series

6.1 Fourier series

Consider a periodic function $s(t)$ that repeats outside the interval $t \in [-L, L]$ and satisfies:

1. $\int_{-L}^{L} |s(t)|\, dt < \infty$,
2. $s(t)$ has a finite number of discontinuities,
3. $s(t)$ has a finite number of extrema.



Note: $s(t)$ could also just be defined only on $[-L, L]$.

Then it is possible to write $s(t)$ as an infinite sum of sine and cosine functions:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{\pi n t}{L}\right) + b_n \sin\left(\frac{\pi n t}{L}\right) \right] \tag{6.1}$$

Idea: the coefficients (a_n, b_n) are like the "components" of an infinite-dimensional "vector", where the orthonormal "basis" elements are sines and cosines. For example, a vector in three dimensions is $\vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath} + v_z\,\hat{k}$:

$$(v_x, v_y, v_z) \;\to\; (a_n, b_n), \qquad (\hat{\imath}, \hat{\jmath}, \hat{k}) \;\to\; (\sin, \cos).$$

Orthogonality

Here the inner product is now an integral over $[-L, L]$. For vectors, $\langle \vec{v}, \vec{u} \rangle = \sum_i v_i u_i$; for functions

$$\langle s(t), r(t) \rangle = \frac{1}{L} \int_{-L}^{L} s(t)\, r(t)\, dt\,.$$

For the unit vectors, $\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1$ and $\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{k}=\hat{\imath}\cdot\hat{k}=0$. The sines and cosines satisfy the same relations, for integers $m,n\geq 1$:

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\sin\left(\frac{m\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)=\delta_{mn}} \tag{6.2}$$

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\cos\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)=\delta_{mn}} \tag{6.3}$$

$$\boxed{\frac{1}{L}\int_{-L}^{L}dt\,\sin\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)=0} \tag{6.4}$$

Check. Use $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$. For $m\neq n$ both cosines integrate to zero over a whole number of periods; for $m=n$ the first term is 1 and the second integrates to zero, so the integral is $\frac{1}{L}\cdot\frac{1}{2}\cdot 2L=1$. The cosine–cosine case is identical with $\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$, and the sine–cosine integrand is odd, so it integrates to zero over the symmetric interval.

The coefficients

Multiply (6.1) by $\cos(m\pi t/L)$ and integrate. For $m\neq 0$, using (6.3) and (6.4):

$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)\cos\left(\frac{m\pi t}{L}\right)=\frac{1}{L}\int_{-L}^{L}dt\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right]\cos\left(\frac{m\pi t}{L}\right)=\sum_{n=1}^{\infty}(a_n\delta_{mn}+0)=a_m\,.$$

For $m=0$:

$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)=\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\,dt=\frac{a_0}{2}\cdot\frac{2L}{L}=a_0\,.$$

Similarly, using (6.2) and (6.4),

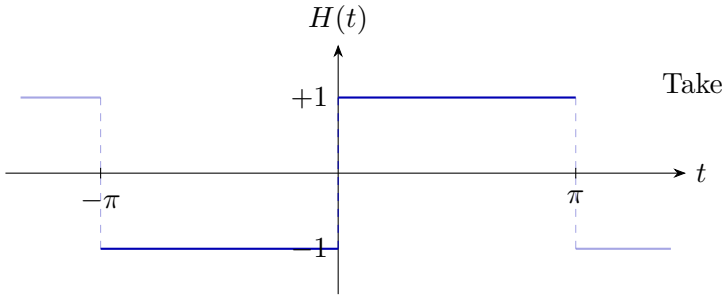
$$\frac{1}{L}\int_{-L}^{L}dt\,s(t)\sin\left(\frac{m\pi t}{L}\right)=0+\sum_{n=1}^{\infty}(0+b_n\delta_{mn})=b_m\,.$$

Collecting these results:

$$\boxed{a_n=\frac{1}{L}\int_{-L}^{L}dt\,s(t)\cos\left(\frac{n\pi t}{L}\right),\quad n\geq 0} \tag{6.5}$$

$$\boxed{b_n=\frac{1}{L}\int_{-L}^{L}dt\,s(t)\sin\left(\frac{n\pi t}{L}\right),\quad n\geq 1} \tag{6.6}$$

6.2 Example: the step function



Take $L=\pi$ and $s(t)=H(t)$, where

$$H(t)=\begin{cases}-1, & -\pi<t<0,\\ +1, & 0<t<\pi.\end{cases}$$

Since H is odd and $\cos(nt)$ is even, the integrand of a_n is odd and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, H(t)\cos(nt) = 0\,.$$

For b_n the integrand is even (odd $\times$ odd), so we can double the integral over $[0,\pi]$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, H(t)\sin(nt) = \frac{2}{\pi}\int_0^{\pi} dt\, \sin(nt) = -\frac{2}{\pi}\,\frac{\cos(nt)}{n}\Big|_0^{\pi} = \frac{2}{n\pi}\,(1-(-1)^n)\,,$$

using $\cos(n\pi) = +1$ for even n and -1 for odd n, and $\cos 0 = 1$. Hence

$$b_n = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd},\\[2mm] 0, & n \text{ even}, \end{cases} \qquad \text{so} \qquad H(t) = \frac{4}{\pi}\left[\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right].$$

6.3 From Fourier series to the Fourier transform

The Fourier series (6.1) describes a function that repeats with period $2L$. A signal that does not repeat, such as a single pulse, can be thought of as the limit $L\to\infty$ of a periodic one. Let us see what happens to the series in this limit.

Complex form

Write $\omega_n = n\pi/L$ for the frequency of the nth term. Then (6.1), (6.5) and (6.6) read

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right], \qquad \begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\cos(\omega_n t'),\\ b_n &= \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\sin(\omega_n t'). \end{aligned}$$

The neighbouring frequencies are separated by $\Delta\omega = \omega_{n+1}-\omega_n = \pi/L$. Combine the two coefficients into one complex number,

$$F_n \equiv a_n - i\,b_n = \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\left[\cos(\omega_n t') - i\sin(\omega_n t')\right] = \frac{1}{L}\int_{-L}^{L} dt'\, s(t')\, e^{-i\omega_n t'},$$

where we used Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Since $(a_n - ib_n)\,e^{i\omega_n t} = a_n\cos(\omega_n t) + b_n\sin(\omega_n t) + i\,[a_n\sin(\omega_n t) - b_n\cos(\omega_n t)]$, each term of the series is the real part of $F_n e^{i\omega_n t}$:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\mathrm{Re}\big[F_n\, e^{i\omega_n t}\big]\,.$$

The limit $L\to\infty$

As L grows, the coefficients F_n shrink like $1/L$, so we define the rescaled quantity $F(\omega_n)\equiv L\,F_n$, which stays finite. Using $1/L = \Delta\omega/\pi$ the series becomes

$$s(t) = \frac{a_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\mathrm{Re}\big[F(\omega_n)\, e^{i\omega_n t}\big]\,\Delta\omega\,, \qquad F(\omega_n) = \int_{-L}^{L} dt'\, s(t')\, e^{-i\omega_n t'}.$$

When $L\to\infty$:

- the spacing $\Delta\omega = \pi/L \to 0$, so the allowed frequencies $\omega_n = n\Delta\omega$ fill out a continuous variable ω and the sum over n becomes an integral over $\omega\in(0,\infty)$;

- $a_0/2 = \frac{1}{2L}\int s\,dt \to 0$, because $\int |s|\,dt$ is finite.

We therefore obtain the **Fourier transform** of $s(t)$,

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} dt\, s(t)\, e^{-i\omega t}} \tag{6.7}$$

and the representation of the signal

$$s(t) = \frac{1}{\pi}\,\mathrm{Re}\int_0^{\infty} d\omega\, F(\omega)\, e^{i\omega t}.$$

The inverse transform

Since $s(t)$ is real, complex conjugating (6.7) gives $F(\omega)^\star = F(-\omega)$. The complex conjugate of the integrand $F(\omega)e^{i\omega t}$ is therefore $F(-\omega)e^{-i\omega t}$, and $\mathrm{Re}\, z = \frac{1}{2}(z + z^\star)$ gives

$$\mathrm{Re}\int_0^{\infty} d\omega\, F(\omega)e^{i\omega t} = \frac{1}{2}\int_0^{\infty} d\omega\, F(\omega)e^{i\omega t} + \frac{1}{2}\int_0^{\infty} d\omega\, F(-\omega)e^{-i\omega t} = \frac{1}{2}\int_{-\infty}^{\infty} d\omega\, F(\omega)e^{i\omega t},$$

where the second term was rewritten by substituting $\omega \to -\omega$. Hence

$$\boxed{s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{i\omega t}} \tag{6.8}$$

Compare this with the Fourier series: the discrete frequencies ω_n and coefficients (a_n, b_n) have been replaced by a continuous frequency ω and the complex function $F(\omega)$, whose real part plays the role of a_n and minus its imaginary part that of b_n (up to the overall factor L). The signal conditions are the same as for the series, with $\int_{-\infty}^{\infty} |s(t)|\,dt < \infty$.

Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

Fourier problem set

These problems are from the course (not from the sources above).

1 (a) Plot the waveform of the periodic function defined as

$$f(t) = -\frac{2t}{\pi} \quad \text{if} \quad -\pi \le t \le \pi, \qquad f(t) = f(t + 2\pi) \quad \text{for all } t\,.$$

(b) Compute the Fourier series of this function for all n.

[Answer →]

2 (a) Using the provided python code `fourier_sawtooth.py`, plot the 'sawtooth' waveform as a function of time for a value of w not equal to zero, one or 1/2. Try to pick a value that will highlight interesting features of the Fourier series and explain why you made your choice. What is special about the values $w = 0$, $w = 1/2$ and $w = 1$?

(b) For the value of w chosen above, plot the Fourier coefficients for up to $N = 15$.

(c) Choose 3 values of N (different from 15) for which to plot the N^{th} order Fourier approximation that illustrate different characteristics of the approximation. Explain what is happening as N gets large and why you chose the values you did.

[Answer →]

3 Modify `fourier_sawtooth.py` to check your answer to problem 1. Make a plot to compare the analytic solution obtained to the numerical computation (by plotting the difference, plotting them side-by-side, or some other way).

[Answer →]

Answers

Chapter 6: Fourier Series

1 (b) $a_n = 0$ for all n; $b_n = \dfrac{4(-1)^n}{n\pi}$ for $n \geq 1$ [← Problem]

2 (a) $w = 0$ and $w = 1$ give a falling and a rising linear ramp; $w = 1/2$ gives a symmetric triangle wave [← Problem]

3 The numerical b_n agree with $4(-1)^n/(n\pi)$ and the numerical a_n vanish (up to numerical error) [← Problem]

Sources and Attribution

The end-of-chapter problems in these notes are taken from the following openly licensed textbooks. Problem numbers in the text (for example 5.53) are the numbers of the end-of-chapter problems in the book named in the problems section.

- OpenStax, *University Physics Volume 2*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-2
- OpenStax, *University Physics Volume 1*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-1
- OpenStax, *Calculus Volume 3*, by Gilbert Strang, Edwin "Jed" Herman et al. Licensed under CC BY 4.0. https://openstax.org/details/books/calculus-volume-3

The signal-processing chapter also draws on the following book, which is not openly licensed and is cited rather than reproduced: Marina Bosi and Richard E. Goldberg, *Introduction to Digital Audio Coding and Standards*, Kluwer Academic Publishers, 2003. The approximation of the threshold in quiet used there is taken from that book; the plot of it is our own.

The Fourier problems at the end of the Fourier chapter were written for this course by the author.

The licence is Creative Commons Attribution 4.0 International (https://creativecommons.org/licenses/by/4.0/). Changes: figures have been redrawn, and the wording of some problems has been lightly edited (for example "as shown below" → "as shown"), and the brief answers are taken from the answer keys provided by OpenStax. The original problems, figures and answers are available at the sources above. These notes are not endorsed by OpenStax.